Control policies for 421, a stochastic game of life

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Abstract

We treat a stochastic control problem, inspired by a popular dice game, known as 421 [1]. Tools are realized, especially for combination algebra and quotient algebra modulo face permutations. Linear growth Markovian fate trees are evaluated. A Markovian utility-strategy format is introduced, along with purifying, forcing and pruning techniques. Backward induction programs are realized: - an optimal policy, `mean-max`, - a Markovian strategy judging program, `mean-mean`. The latter is also used, numerically, to evaluate probabilities, by solving Kolmogorov equation, and, symbolically, to display strategies. A constant-goal ratchet stratagem is found and most probably successful strategy final state probabilities are compiled, using dynamic self-similarity. Goal-driven policies (cheaper than `mean-max`) are inferred, depending on compiled probabilities and three parameters: *serendipity* [2], horizon and dynamism. Eight goal-driven policies are applied to seventeen utility functions and most resulting stategies, not all Markovian, are exactly judged. Empiric laws of policy utility are inferred, confirming the utility of serendipity [2]. Meta-policy is introduced. Technically, *Mathematica* allows choosing between the λ (anonymous), functional and procedural programming styles, depending on evaluation and naming constraints.

Key words: utility, strategy, policy, insufficient reason, backward induction, goal, ratchet, serendipity, horizon, dynamism, meta-policy.

1. Introduction

1.1. Playing against providence

Within the 421 game, which is fully described in [1], only a round is treated (when one player alone is opposed to providence). As providence is not exactly a player, a round is not exactly a game, but a stochastic control problem. A *combination* is a list, the order of which does not matter, or a finite set possibly with repetitions. A *state* is a combination of dice that have been irreversibly pushed away from the dice board; an *event* is a combination of dice that have just been cast. The cast number (initially zero) serves as a discrete date. After each event, a new state is chosen. *Fate* is an alternate sequence state-event-state… ending with some state having some utility, which may depend on final date and state but not on history. A set of possible fates, along with event probabilities and final utilities, appears as a *fate tree*.

A *strategy* is a (generally probabilistic) determination of all possible choices. A *policy* is a program evaluating exactly one strategy. Distinct policies can evaluate the same strategy.

The *Von Neumann-Morgenstern* or "`mean-max`" *theorem* [3]:
- utility exactly expresses preference, that is, choice maximizes utility,
- the utility of present state is the expected utility of next random events.

If many states are most useful (as in Buridan's donkey story), then, according to the principle of insufficient reason [1, 4], completely *equiprobale* choice (among all most useful states) is assumed. From this and the von Neumann-Morgenstern theorem, there exists exactly one complete most useful strategy, non-pure in general and evaluatable by "backward induction" [3].

As dice are independent and, by insufficient reason, unloaded, the probabilities depending on face combinations are invariant by global face permutations.

There are d dice each with f faces. A player casts the dice at most j times. The parameters of the control problem are thus d, f, j and some utility function (a functional parameter). Normally,

```
{d = 3, f = 6, j = 3}
```

although next players begin to play with smaller j. Moreover, smaller d, f, j, like

```
{d = 2, f = 3, j = 3}
```

provide small-scale cases, useful for program development.

1.2. Conventions and data

« ⚠ » introduces agenda (things to do), « ⚠ » introduces some hopefully clever remarks.

In text, *loose* sign conventions are used, that is,
- ≥ is greater, > is strictly greater,
- zero is both positive and negative,
- a constant function is both increasing and decreasing,
- present is both past and future,
…

```
$Version
```

```
5.2 for Mac OS X (June 20, 2005)
```

After initializing, every *numbered* section of the present notebook can be evaluated independently in a meaningful way. A section usually consists of three parts: one for developing, one for defining and one for testing some function. Keep the environment clean by evaluating clearing commands as well!

Parameters will be often grouped as in

```
action_[conditions__]@object__
```

This grouping simplifies mapping, as in

```
Block[{power1}, power1[n_]@x_ := Power[x, n];
 Print[Power[#, 3] & /@ Range@3];
 Print[(*simpler syntax:*)power1@3 /@ Range@3]
]
```

```
{1, 8, 27}

{1, 8, 27}
```

make (like the unix command) is used to hide parameters that are too big to show (or that would take too much time to evaluate from a compact form). **make** is not supposed to evaluate anything, but to affect variables: this is procedural programming and it appears as a consequence of evaluation constraints. ⚠ If **this** already has a value, it will be affected instead of **this**.

```
make[implicit___]@this[explicit__] := this@explicit = function1[implicit, explicit]
```

Probability and strategy utility files should accompany the present notebook. Otherwise, probability files will be automatically regenerated when needed; the strategy utility files can be regenerated by evaluating section 7.4 (not so quietly because of some hard cases).

```
<< AuthorTools`
SetDirectory@NotebookFolder@InputNotebook[];

FileNames[{"probability*", "*trategyUtility*"}]
```

```
{maxStrategyUtility.txt, probability1_2_3.txt, probability2_3_3.txt,
 probability2_3_4.txt, probability2_4_3.txt, probability3_6_3.txt, probability3_6_3V.txt,
 randomStrategyUtility.txt, strategyUtility1.txt, strategyUtility.txt}
```

⚠ This beeps once for unknown reason but works all the same.

```
Utilities`Notation`AutoLoadNotationPalette = False;
<< Utilities`Notation`
```

2. Tools

2.1. Evaluation constraints

2.1.1. Timing

2.1.2. Spacing

2.2. Lists

2.2.1. Only element filtering

2.2.2. Elementary statistics

2.2.3. Reverse folding and history

2.2.4. Function determined by graph

2.2.5. ArgMax

2.2.6. Quotient of list modulo function

2.2.7. Filling table from coordinates and values

2.2.8. Slicing

2.3. Formats

2.3.1. Long table slicing

2.3.2. Boxes

2.3.3. Line labeling

3. Combinations

3.1. Generalizing finite set algebra

3.1.1. All combinations

3.1.2. Eulerian (occupation number) vector form

3.1.3. Element query

3.1.4. Difference

3.1.5. Intersection

3.1.6. Inclusion query

3.2. Equivalence modulo face permutations

3.2.1. Face permutation

3.2.2. Combination representative

3.2.3. Representing permutation

3.2.4. All representatives

3.2.5. Conditional representative

3.2.6. Birepresentative

3.2.7. All conditional representatives and all birepresentatives

4. Fate, utility and strategy

4.1. Linear growth Markovian fate tree

4.1.1. Branching one leaf

4.1.2. Branching many leaves and sharing next states

4.1.3. Branching deepest level

4.1.4. Branching completely

```
{d = 2, f = 2, j = 3}; restrictNextStates = Identity;
branch[d, f, restrictNextStates][fateTree → {{0, {{}}}}]
```

```
{{0, {{{{}, {{{1/4, {1, 1}}, {{{}, {1}}, {{1}, {2}}, {{1, 1}, {4}}}},
      {{1/2, {1, 2}}, {{{}, {1}}, {{1}, {2}}, {{2}, {3}}, {{1, 2}, {5}}}},
      {{1/4, {2, 2}}, {{{}, {1}}, {{2}, {3}}, {{2, 2}, {6}}}}}}}, {}}},
 {1, {{}, {1}, {2}, {1, 1}, {1, 2}, {2, 2}}}}
```

```
Fold[branch[d, f, #2][fateTree → #1] &, {{0, {{}}}},
 Join[Table[restrictNextStates, {j - 1}], {takeLast, takeLast}]]
```

```
{{0, {{{{}, {{{1/4, {1, 1}}, {{{}, {1}}, {{1}, {2}}, {{1, 1}, {4}}}},
      {{1/2, {1, 2}}, {{{}, {1}}, {{1}, {2}}, {{2}, {3}}, {{1, 2}, {5}}}},
      {{1/4, {2, 2}}, {{{}, {1}}, {{2}, {3}}, {{2, 2}, {6}}}}}}}, {}}},
 {1, {{{{}, {{{1/4, {1, 1}}, {{{}, {1}}, {{1}, {2}}, {{1, 1}, {4}}}},
      {{1/2, {1, 2}}, {{{}, {1}}, {{1}, {2}}, {{2}, {3}}, {{1, 2}, {5}}}},
      {{1/4, {2, 2}}, {{{}, {1}}, {{2}, {3}}, {{2, 2}, {6}}}}}}},
    {{1}, {{{1/2, {1}}, {{{1}, {2}}, {{1, 1}, {4}}}}, {{1/2, {2}}, {{{1}, {2}}, {{1, 2}, {5}}}}}}},
    {{2}, {{{1/2, {1}}, {{{2}, {3}}, {{1, 2}, {5}}}}, {{1/2, {2}}, {{{2}, {3}}, {{2, 2}, {6}}}}}}}},
   {{1, 1}, {1, 2}, {2, 2}}}}, {2, {{{{}, {{{1/4, {1, 1}}, {{{1, 1}, {1}}}},
      {{1/2, {1, 2}}, {{{1, 2}, {2}}}}, {{1/4, {2, 2}}, {{{2, 2}, {3}}}}}}},
    {{1}, {{{1/2, {1}}, {{{1, 1}, {1}}}}, {{1/2, {2}}, {{{1, 2}, {2}}}}}}},
    {{2}, {{{1/2, {1}}, {{{1, 2}, {2}}}}, {{1/2, {2}}, {{{2, 2}, {3}}}}}}}},
   {{1, 1}, {1, 2}, {2, 2}}}}, {3, {{}, {{1, 1}, {1, 2}, {2, 2}}}}, {4, {}}}
```

```
fateTree1 = dropLast@Fold[branch[d, f, #2][fateTree → #1] &,
   {{0, {{}}}}, Join[Table[restrictNextStates, {j - 1}], {takeLast, takeLast}]]
```

```
{{0, {{{{}, {{{1/4, {1, 1}}, {{{}, {1}}, {{1}, {2}}, {{1, 1}, {4}}}},
      {{1/2, {1, 2}}, {{{}, {1}}, {{1}, {2}}, {{2}, {3}}, {{1, 2}, {5}}}},
      {{1/4, {2, 2}}, {{{}, {1}}, {{2}, {3}}, {{2, 2}, {6}}}}}}}, {}}},
 {1, {{{{}, {{{1/4, {1, 1}}, {{{}, {1}}, {{1}, {2}}, {{1, 1}, {4}}}},
      {{1/2, {1, 2}}, {{{}, {1}}, {{1}, {2}}, {{2}, {3}}, {{1, 2}, {5}}}},
      {{1/4, {2, 2}}, {{{}, {1}}, {{2}, {3}}, {{2, 2}, {6}}}}}}},
    {{1}, {{{1/2, {1}}, {{{1}, {2}}, {{1, 1}, {4}}}}, {{1/2, {2}}, {{{1}, {2}}, {{1, 2}, {5}}}}}}},
    {{2}, {{{1/2, {1}}, {{{2}, {3}}, {{1, 2}, {5}}}}, {{1/2, {2}}, {{{2}, {3}}, {{2, 2}, {6}}}}}}}},
   {{1, 1}, {1, 2}, {2, 2}}}}, {2, {{{{}, {{{1/4, {1, 1}}, {{{1, 1}, {1}}}},
      {{1/2, {1, 2}}, {{{1, 2}, {2}}}}, {{1/4, {2, 2}}, {{{2, 2}, {3}}}}}}},
    {{1}, {{{1/2, {1}}, {{{1, 1}, {1}}}}, {{1/2, {2}}, {{{1, 2}, {2}}}}}}},
    {{2}, {{{1/2, {1}}, {{{1, 2}, {2}}}}, {{1/2, {2}}, {{{2, 2}, {3}}}}}}}},
   {{1, 1}, {1, 2}, {2, 2}}}}, {3, {{}, {{1, 1}, {1, 2}, {2, 2}}}}}
```

```
{restrictNextStates, fateTree1} =.; {d =., f =., j =.};
```

```
Unprotect@fateTree; Clear@fateTree;
fateTree[player_, d_, f_, j_Integer] /; j > 0 :=
 dropLast@Fold[branch[d, f, #2][fateTree → #1] &, {{0, {{}}}}, Join[
    Switch[player,
     "p1", Table[Identity, {j - 1}],
     "p2", Table[dropLast, {j - 1}]], {takeLast, takeLast}]]
Protect@fateTree;
```

```
fateTree["p1", 0, 1, 2]
```

```
{{0, {{}, {{}}}}, {1, {{}, {}}}, {2, {{}, {}}}}
```

```
fateTree["p1", 1, 0, 1]
```

```
{{0, {{{{}, {}}}, {}}}, {1, {{}, {}}}}
```

```
fateTree["p2", 1, 1, 0]
```

```
fateTree[p2, 1, 1, 0]
```

```
fateTree["p2", 0, 1, 1]
```

```
{{0, {{}, {{}}}}, {1, {{}, {}}}}
```

```
fateTree["p2", 1, 1, 1]
```

```
{{0, {{{{}, {{{1, {1}}, {{{1}, {1}}}}}}}, {}}}, {1, {{}, {{1}}}}}
```

```
fateTree["p2", 1, 1, 2]
```

```
{{0, {{{{}, {{{1, {1}}, {{{}, {1}}}}}}}, {}}},
 {1, {{{{}, {{{1, {1}}, {{{1}, {1}}}}}}}, {}}}, {2, {{}, {{1}}}}}
```

```
fateTree["p2", 2, 1, 3]
```

```
{{0, {{{{}, {{{1, {1, 1}}, {{{}, {1}}, {{1}, {2}}}}}}}, {}}},
 {1, {{{{}, {{{1, {1, 1}}, {{{}, {1}}, {{1}, {2}}}}}}, {{1}, {{{1, {1}}, {{{1}, {2}}}}}}}, {}}},
 {2, {{{{}, {{{1, {1, 1}}, {{{1, 1}, {1}}}}}}, {{1}, {{{1, {1}}, {{{1, 1}, {1}}}}}}}, {}}},
 {3, {{}, {{1, 1}}}}}
```

```
fateTree["p2", 2, 2, 2]
```

```
{{0, {{{{}, {{{1/4, {1, 1}}, {{{}, {1}}, {{1}, {2}}}}}, {{1/2, {1, 2}},
        {{{}, {1}}, {{1}, {2}}, {{2}, {3}}}}}, {{1/4, {2, 2}}, {{{}, {1}}, {{2}, {3}}}}}}}, {}}},
 {1, {{{{}, {{{1/4, {1, 1}}, {{{1, 1}, {1}}}}}, {{1/2, {1, 2}}, {{{1, 2}, {2}}}},
       {{1/4, {2, 2}}, {{{2, 2}, {3}}}}}}},
     {{1}, {{{1/2, {1}}, {{{1, 1}, {1}}}}}, {{1/2, {2}}, {{{1, 2}, {2}}}}}}},
     {{2}, {{{1/2, {1}}, {{{1, 2}, {2}}}}}, {{1/2, {2}}, {{{2, 2}, {3}}}}}}}, {}}},
 {2, {{}, {{1, 1}, {1, 2}, {2, 2}}}}}
```

```
fateTree["p1", 2, 3, 1]
```

```
{{0, {{{{}, {{{1/9, {1, 1}}, {{{1, 1}, {1}}}}}, {{2/9, {1, 2}}, {{{1, 2}, {2}}}},
       {{2/9, {1, 3}}, {{{1, 3}, {3}}}}}, {{1/9, {2, 2}}, {{{2, 2}, {4}}}},
       {{2/9, {2, 3}}, {{{2, 3}, {5}}}}}, {{1/9, {3, 3}}, {{{3, 3}, {6}}}}}}}}, {}}},
 {1, {{}, {{1, 1}, {1, 2}, {1, 3}, {2, 2}, {2, 3}, {3, 3}}}}}
```

⚠ Next state sharing, on one hand, makes fate tree growing only linearly with cast number, not exponentially; on the other hand, prevents remembering fate: fate tree is Markovian.

Normal case

Apple Powerbook G4, PPC 1 GHz

```
{d = 3, f = 6, j = 3}; Timing@ByteCount@fateTree["p1", d, f, j]
```

```
{0.44 Second, 407416}
```

```
spacing[fateTree["p1", d, f, j], Last@%]
```

```
691856
```

```
Timing@ByteCount@fateTree["p2", d, f, j]
```

```
{0.28 Second, 340056}
```

```
{d =., f =., j =.};
```

4.2. Markovian utility and strategy formats

4.2.1. …with utility

Check unicity of initial state.

```
Clear@initial;
initial@{{_, {{{_, {x_, ___}}}, {}}}, ___} = Extract[only@{{{x}}}, {1, 1}];
```

Utility precedes a strategy piece, empty for immediate utility.

```
utilityAndStrategy1 = {{0,
   {{{{}, {-7, {{{1, 1}, {{{1, 1}, {4}}}}, {{1, 2}, {{{1, 2}, {5}}}}, {{2, 2}, {{{2, 2}, {6}}}}}}},
    {}}}, {1,
   {{{{}, {-17, {{{1, 1}, {{{1, 1}, {1}}}}, {{1, 2}, {{{1, 2}, {2}}}}, {{2, 2}, {{{2, 2}, {3}}}}}}},
     {{1}, {-35/2, {{{1}, {{{1, 1}, {1}}}}, {{2}, {{{1, 2}, {2}}}}}}},
     {{2}, {-33/2, {{{1}, {{{1, 2}, {2}}}}, {{2}, {{{2, 2}, {3}}}}}}}},
    {{{1, 1}, {-8, {}}}, {{1, 2}, {-7, {}}}, {{2, 2}, {-6, {}}}}}},
  {2, {{}, {{{1, 1}, {-18, {}}}, {{1, 2}, {-17, {}}}, {{2, 2}, {-16, {}}}}}}};
```

```
initial@utilityAndStrategy1
```

```
-7
```

```
initial@Rest@utilityAndStrategy1
```

```
initial[{{1,
   {{{{}, {-17, {{{1, 1}, {{{1, 1}, {1}}}}, {{1, 2}, {{{1, 2}, {2}}}}, {{2, 2}, {{{2, 2}, {3}}}}}}},
     {{1}, {-35/2, {{{1}, {{{1, 1}, {1}}}}, {{2}, {{{1, 2}, {2}}}}}}},
     {{2}, {-33/2, {{{1}, {{{1, 2}, {2}}}}, {{2}, {{{2, 2}, {3}}}}}}}},
    {{{1, 1}, {-8, {}}}, {{1, 2}, {-7, {}}}, {{2, 2}, {-6, {}}}}}},
  {2, {{}, {{{1, 1}, {-18, {}}}, {{1, 2}, {-17, {}}}, {{2, 2}, {-16, {}}}}}}}]
```

```
utilityAndStrategy1 =.
```

4.2.2. …without utility

```
Clear@extractStrategy; extractStrategy@utilityAndStrategy1_List :=
 MapAt[Map[MapAt[Last, #, -1] &, #, {2}] &, #, -1] & /@ utilityAndStrategy1;
```

```
extractStrategy@{{0, {{{{}, {-7,
         {{{1, 1}, {{{1, 1}, {4}}}}, {{1, 2}, {{{1, 2}, {5}}}}, {{2, 2}, {{{2, 2}, {6}}}}}}}, {}}},
   {1, {{{{}, {-17, {{{1, 1}, {{{1, 1}, {1}}}}, {{1, 2}, {{{1, 2}, {2}}}}, {{2, 2},
            {{{2, 2}, {3}}}}}}}, {{1}, {-35/2, {{{1}, {{{1, 1}, {1}}}}, {{2}, {{{1, 2}, {2}}}}}}},
      {{2}, {-33/2, {{{1}, {{{1, 2}, {2}}}}, {{2}, {{{2, 2}, {3}}}}}}}},
     {{{1, 1}, {-8, {}}}, {{1, 2}, {-7, {}}}, {{2, 2}, {-6, {}}}}}},
   {2, {{}, {{{1, 1}, {-18, {}}}, {{1, 2}, {-17, {}}}, {{2, 2}, {-16, {}}}}}}} == {{0,
   {{{{}, {{{1, 1}, {{{1, 1}, {4}}}}, {{1, 2}, {{{1, 2}, {5}}}}, {{2, 2}, {{{2, 2}, {6}}}}}}, {}}},
  {1, {{{{}, {{{1, 1}, {{{1, 1}, {1}}}}, {{1, 2}, {{{1, 2}, {2}}}}, {{2, 2}, {{{2, 2}, {3}}}}}},
     {{1}, {{{1}, {{{1, 1}, {1}}}}, {{2}, {{{1, 2}, {2}}}}}},
     {{2}, {{{1}, {{{1, 2}, {2}}}}, {{2}, {{{2, 2}, {3}}}}}}},
    {{{1, 1}, {}}, {{1, 2}, {}}, {{2, 2}, {}}}}},
  {2, {{}, {{{1, 1}, {}}, {{1, 2}, {}}, {{2, 2}, {}}}}}}
```

```
True
```

4.2.3. Utility function samples

4.3. Fate tree or strategy transformations

4.3.1. Purifying strategy

4.3.2. Forcing fate tree by some strategy

4.3.3. Pruning fate tree according to some utility function

5. Backward induction programs

5.1. Mean-max policy

5.1.1. Fructifying

5.1.2. Folding one state branch

5.1.3. Folding deepest level

5.1.4. Folding completely

Time scheme.

```
{d = 2, f = 3, j = 3};
With[{fructifiedFateTree = Range[0, j]},
 reverseFoldList[meanMaxStep@{#1, #2} &,
  dropLast@fructifiedFateTree, Last@fructifiedFateTree]]
```

```
{meanMaxStep[{0, meanMaxStep[{1, meanMaxStep[{2, 3}]}]}],
 meanMaxStep[{1, meanMaxStep[{2, 3}]}], meanMaxStep[{2, 3}], 3}
```

```
utility = -10 #1 + Total@#2 &; fructifiedFateTree = fructify[utility]@fateTree["p1", d, f, j];
```

```
reverseFoldList[meanMaxStep@{#1, #2} & , dropLast@fructifiedFateTree, Last@fructifiedFateTree]
```

```
{{0, {{{{}, {-6, {{{1, 1}, {{{1, 1}, {5}}}}, {{1, 2}, {{{1, 2}, {6}}}}, {{1, 3}, {{{1, 3}, {7}}}},
       {{2, 2}, {{{2, 2}, {8}}}}, {{2, 3}, {{{2, 3}, {9}}}}, {{3, 3}, {{{3, 3}, {10}}}}}}}}, {}}},
 {1, {{{{}, {-16, {{{1, 1}, {{{1, 1}, {5}}}}, {{1, 2}, {{{1, 2}, {6}}}}, {{1, 3}, {{{1, 3}, {7}}}},
       {{2, 2}, {{{2, 2}, {8}}}}, {{2, 3}, {{{2, 3}, {9}}}}, {{3, 3}, {{{3, 3}, {10}}}}}}},
    {{1}, {-17, {{{1}, {{{1, 1}, {5}}}}, {{2}, {{{1, 2}, {6}}}}, {{3}, {{{1, 3}, {7}}}}}}},
    {{2}, {-16, {{{1}, {{{1, 2}, {6}}}}, {{2}, {{{2, 2}, {8}}}}, {{3}, {{{2, 3}, {9}}}}}}},
    {{3}, {-15, {{{1}, {{{1, 3}, {7}}}}, {{2}, {{{2, 3}, {9}}}}, {{3}, {{{3, 3}, {10}}}}}}}},
   {{{1, 1}, {-8, {}}}, {{1, 2}, {-7, {}}}, {{1, 3}, {-6, {}}},
    {{2, 2}, {-6, {}}}, {{2, 3}, {-5, {}}}, {{3, 3}, {-4, {}}}}}},
 {2, {{{{}, {-26, {{{1, 1}, {{{1, 1}, {1}}}}, {{1, 2}, {{{1, 2}, {2}}}}, {{1, 3}, {{{1, 3}, {3}}}},
       {{2, 2}, {{{2, 2}, {4}}}}, {{2, 3}, {{{2, 3}, {5}}}}, {{3, 3}, {{{3, 3}, {6}}}}}}},
    {{1}, {-27, {{{1}, {{{1, 1}, {1}}}}, {{2}, {{{1, 2}, {2}}}}, {{3}, {{{1, 3}, {3}}}}}}},
    {{2}, {-26, {{{1}, {{{1, 2}, {2}}}}, {{2}, {{{2, 2}, {4}}}}, {{3}, {{{2, 3}, {5}}}}}}},
    {{3}, {-25, {{{1}, {{{1, 3}, {3}}}}, {{2}, {{{2, 3}, {5}}}}, {{3}, {{{3, 3}, {6}}}}}}}},
   {{{1, 1}, {-18, {}}}, {{1, 2}, {-17, {}}}, {{1, 3}, {-16, {}}},
    {{2, 2}, {-16, {}}}, {{2, 3}, {-15, {}}}, {{3, 3}, {-14, {}}}}}},
 {3, {{}, {{{1, 1}, {-28, {}}}, {{1, 2}, {-27, {}}}, {{1, 3}, {-26, {}}},
    {{2, 2}, {-26, {}}}, {{2, 3}, {-25, {}}}, {{3, 3}, {-24, {}}}}}}}
```

```
{d, f, j, fructifiedFateTree, utility} =.;
```

```
Clear@maxUtilityAndStrategy;
maxUtilityAndStrategy[fateTree1_, utility_] :=
 With[{fructifiedFateTree = fructify[utility]@fateTree1},
  reverseFoldList[meanMaxStep@{#1, #2} &,
   dropLast@fructifiedFateTree, Last@fructifiedFateTree]
 ]

maxUtilityAndStrategy[fateTree["p1", 2, 3, 3], -10 #1 + Total@#2 &] ==
 {{0, {{{{}, {-6, {{{1, 1}, {{{1, 1}, {5}}}}, {{1, 2}, {{{1, 2}, {6}}}}, {{1, 3}, {{{1, 3}, {7}}}},
        {{2, 2}, {{{2, 2}, {8}}}}, {{2, 3}, {{{2, 3}, {9}}}}, {{3, 3}, {{{3, 3}, {10}}}}}}}}, {}}},
  {1, {{{{}, {-16, {{{1, 1}, {{{1, 1}, {5}}}}, {{1, 2}, {{{1, 2}, {6}}}}, {{1, 3}, {{{1, 3}, {7}}}},
        {{2, 2}, {{{2, 2}, {8}}}}, {{2, 3}, {{{2, 3}, {9}}}}, {{3, 3}, {{{3, 3}, {10}}}}}}},
     {{1}, {-17, {{{1}, {{{1, 1}, {5}}}}, {{2}, {{{1, 2}, {6}}}}, {{3}, {{{1, 3}, {7}}}}}}},
     {{2}, {-16, {{{1}, {{{1, 2}, {6}}}}, {{2}, {{{2, 2}, {8}}}}, {{3}, {{{2, 3}, {9}}}}}}},
     {{3}, {-15, {{{1}, {{{1, 3}, {7}}}}, {{2}, {{{2, 3}, {9}}}}, {{3}, {{{3, 3}, {10}}}}}}}},
    {{{1, 1}, {-8, {}}}, {{1, 2}, {-7, {}}}, {{1, 3}, {-6, {}}},
     {{2, 2}, {-6, {}}}, {{2, 3}, {-5, {}}}, {{3, 3}, {-4, {}}}}}},
  {2, {{{{}, {-26, {{{1, 1}, {{{1, 1}, {1}}}}, {{1, 2}, {{{1, 2}, {2}}}}, {{1, 3}, {{{1, 3}, {3}}}},
        {{2, 2}, {{{2, 2}, {4}}}}, {{2, 3}, {{{2, 3}, {5}}}}, {{3, 3}, {{{3, 3}, {6}}}}}}},
     {{1}, {-27, {{{1}, {{{1, 1}, {1}}}}, {{2}, {{{1, 2}, {2}}}}, {{3}, {{{1, 3}, {3}}}}}}},
     {{2}, {-26, {{{1}, {{{1, 2}, {2}}}}, {{2}, {{{2, 2}, {4}}}}, {{3}, {{{2, 3}, {5}}}}}}},
     {{3}, {-25, {{{1}, {{{1, 3}, {3}}}}, {{2}, {{{2, 3}, {5}}}}, {{3}, {{{3, 3}, {6}}}}}}}},
    {{{1, 1}, {-18, {}}}, {{1, 2}, {-17, {}}}, {{1, 3}, {-16, {}}},
     {{2, 2}, {-16, {}}}, {{2, 3}, {-15, {}}}, {{3, 3}, {-14, {}}}}}},
  {3, {{}, {{{1, 1}, {-28, {}}}, {{1, 2}, {-27, {}}}, {{1, 3}, {-26, {}}},
     {{2, 2}, {-26, {}}}, {{2, 3}, {-25, {}}}, {{3, 3}, {-24, {}}}}}}}

True

initial@maxUtilityAndStrategy[fateTree["p1", 2, 3, 3], -10 #1 + Total@#2 &]

-6

extractStrategy@maxUtilityAndStrategy[fateTree["p1", 2, 3, 3], -10 #1 + Total@#2 &] ==
 {{0, {{{{}, {{{1, 1}, {{{1, 1}, {5}}}}, {{1, 2}, {{{1, 2}, {6}}}}, {{1, 3}, {{{1, 3}, {7}}}},
       {{2, 2}, {{{2, 2}, {8}}}}, {{2, 3}, {{{2, 3}, {9}}}}, {{3, 3}, {{{3, 3}, {10}}}}}}}, {}}},
  {1, {{{{}, {{{1, 1}, {{{1, 1}, {5}}}}, {{1, 2}, {{{1, 2}, {6}}}}, {{1, 3}, {{{1, 3}, {7}}}},
       {{2, 2}, {{{2, 2}, {8}}}}, {{2, 3}, {{{2, 3}, {9}}}}, {{3, 3}, {{{3, 3}, {10}}}}}},
     {{1}, {{{1}, {{{1, 1}, {5}}}}, {{2}, {{{1, 2}, {6}}}}, {{3}, {{{1, 3}, {7}}}}}},
     {{2}, {{{1}, {{{1, 2}, {6}}}}, {{2}, {{{2, 2}, {8}}}}, {{3}, {{{2, 3}, {9}}}}}},
     {{3}, {{{1}, {{{1, 3}, {7}}}}, {{2}, {{{2, 3}, {9}}}}, {{3}, {{{3, 3}, {10}}}}}}},
    {{{1, 1}, {}}, {{1, 2}, {}}, {{1, 3}, {}}, {{2, 2}, {}}, {{2, 3}, {}}, {{3, 3}, {}}}}},
  {2, {{{{}, {{{1, 1}, {{{1, 1}, {1}}}}, {{1, 2}, {{{1, 2}, {2}}}}, {{1, 3}, {{{1, 3}, {3}}}},
       {{2, 2}, {{{2, 2}, {4}}}}, {{2, 3}, {{{2, 3}, {5}}}}, {{3, 3}, {{{3, 3}, {6}}}}}},
     {{1}, {{{1}, {{{1, 1}, {1}}}}, {{2}, {{{1, 2}, {2}}}}, {{3}, {{{1, 3}, {3}}}}}},
     {{2}, {{{1}, {{{1, 2}, {2}}}}, {{2}, {{{2, 2}, {4}}}}, {{3}, {{{2, 3}, {5}}}}}},
     {{3}, {{{1}, {{{1, 3}, {3}}}}, {{2}, {{{2, 3}, {5}}}}, {{3}, {{{3, 3}, {6}}}}}}},
    {{{1, 1}, {}}, {{1, 2}, {}}, {{1, 3}, {}}, {{2, 2}, {}}, {{2, 3}, {}}, {{3, 3}, {}}}}},
  {3, {{}, {{{1, 1}, {}}, {{1, 2}, {}}, {{1, 3}, {}}, {{2, 2}, {}}, {{2, 3}, {}}, {{3, 3}, {}}}}}}

True
```

5.2. Mean-mean Markovian strategy judging program

5.2.1. Folding one strategy state branch

5.2.2. Folding last step

5.2.3. Folding completely

Time scheme. ⚠ **meanMeanStep** must format utilities as in fate tree last level.

```
{d = 2, f = 4, j = 3};
With[{fructifiedFateTree = t /@ Range[0, j], strategy1 = s /@ Range[0, j]},
 scanPrint@reverseFoldList[meanMeanStep[{First@#1, #2}, Last@#1] &,
   Transpose[dropLast /@ {fructifiedFateTree, strategy1}], Last@fructifiedFateTree]]
```

```
meanMeanStep[{t[0], meanMeanStep[{t[1], meanMeanStep[{t[2], t[3]}, s[2]]}, s[1]]}, s[0]]

meanMeanStep[{t[1], meanMeanStep[{t[2], t[3]}, s[2]]}, s[1]]

meanMeanStep[{t[2], t[3]}, s[2]]

t[3]
```

```
j = 2;
strategy1 = extractStrategy@maxUtilityAndStrategy[fateTree["p1", d, f, j], δ[{1, 2}, #2] &];
fructifiedFateTree = fructify[δ[{3, 4}, #2] &]@fateTree["p1", d, f, j];

{d =., f =., j =.};

reverseFoldList[meanMeanStep[{First@#1, #2}, Last@#1, withStrategy → True] &,
 Transpose[dropLast /@ {fructifiedFateTree, strategy1}], Last@fructifiedFateTree]
```

```
{{0,
  {{{{}, {1/32, {{{1, 1}, {{{1}, {2}}}}, {{1, 2}, {{{1, 2}, {7}}}}, {{1, 3}, {{{1}, {2}}}}, {{1, 4},
        {{{1}, {2}}}}, {{2, 2}, {{{2}, {3}}}}, {{2, 3}, {{{2}, {3}}}}, {{2, 4}, {{{2}, {3}}}},
       {{3, 3}, {{{}, {1}}}}, {{3, 4}, {{{}, {1}}}}, {{4, 4}, {{{}, {1}}}}}}}}, {}}},
 {1, {{{{}, {1/8, {{{1, 1}, {{{1, 1}, {1}}}}, {{1, 2}, {{{1, 2}, {2}}}},
       {{1, 3}, {{{1, 3}, {3}}}}, {{1, 4}, {{{1, 4}, {4}}}}, {{2, 2}, {{{2, 2}, {5}}}},
       {{2, 3}, {{{2, 3}, {6}}}}, {{2, 4}, {{{2, 4}, {7}}}}, {{3, 3}, {{{3, 3}, {8}}}},
       {{3, 4}, {{{3, 4}, {9}}}}, {{4, 4}, {{{4, 4}, {10}}}}}}},
    {{1}, {0, {{{1}, {{{1, 1}, {1}}}}, {{2}, {{{1, 2}, {2}}}}, {{3}, {{{1, 3}, {3}}}},
       {{4}, {{{1, 4}, {4}}}}}}}, {{2}, {0, {{{1}, {{{1, 2}, {2}}}},
       {{2}, {{{2, 2}, {5}}}}, {{3}, {{{2, 3}, {6}}}}, {{4}, {{{2, 4}, {7}}}}}}},
    {{3}, {1/4, {{{1}, {{{1, 3}, {3}}}}, {{2}, {{{2, 3}, {6}}}}, {{3}, {{{3, 3}, {8}}}},
       {{4}, {{{3, 4}, {9}}}}}}}, {{4}, {1/4, {{{1}, {{{1, 4}, {4}}}},
       {{2}, {{{2, 4}, {7}}}}, {{3}, {{{3, 4}, {9}}}}, {{4}, {{{4, 4}, {10}}}}}}}}},
   {{{1, 1}, {0, {}}}, {{1, 2}, {0, {}}}, {{1, 3}, {0, {}}}, {{1, 4}, {0, {}}},
    {{2, 2}, {0, {}}}, {{2, 3}, {0, {}}}, {{2, 4}, {0, {}}},
    {{3, 3}, {0, {}}}, {{3, 4}, {1, {}}}, {{4, 4}, {0, {}}}}}},
 {2, {{}, {{{1, 1}, {0, {}}}, {{1, 2}, {0, {}}}, {{1, 3}, {0, {}}}, {{1, 4}, {0, {}}},
    {{2, 2}, {0, {}}}, {{2, 3}, {0, {}}}, {{2, 4}, {0, {}}},
    {{3, 3}, {0, {}}}, {{3, 4}, {1, {}}}, {{4, 4}, {0, {}}}}}}}
```

```
meanMeanStep[{fructifiedFateTree[[2]], fructifiedFateTree[[3]]},
 strategy1[[2]], withStrategy → True]
```

```
{1, {{{{}, {1/8, {{{1, 1}, {{{1, 1}, {1}}}},
      {{1, 2}, {{{1, 2}, {2}}}}, {{1, 3}, {{{1, 3}, {3}}}}, {{1, 4}, {{{1, 4}, {4}}}},
      {{2, 2}, {{{2, 2}, {5}}}}, {{2, 3}, {{{2, 3}, {6}}}}, {{2, 4}, {{{2, 4}, {7}}}},
      {{3, 3}, {{{3, 3}, {8}}}}, {{3, 4}, {{{3, 4}, {9}}}}, {{4, 4}, {{{4, 4}, {10}}}}}}},
   {{1}, {0, {{{1}, {{{1, 1}, {1}}}}, {{2}, {{{1, 2}, {2}}}}, {{3}, {{{1, 3}, {3}}}},
      {{4}, {{{1, 4}, {4}}}}}}}, {{2}, {0, {{{1}, {{{1, 2}, {2}}}},
      {{2}, {{{2, 2}, {5}}}}, {{3}, {{{2, 3}, {6}}}}, {{4}, {{{2, 4}, {7}}}}}}},
   {{3}, {1/4, {{{1}, {{{1, 3}, {3}}}}, {{2}, {{{2, 3}, {6}}}}, {{3}, {{{3, 3}, {8}}}},
      {{4}, {{{3, 4}, {9}}}}}}}, {{4}, {1/4, {{{1}, {{{1, 4}, {4}}}},
      {{2}, {{{2, 4}, {7}}}}, {{3}, {{{3, 4}, {9}}}}, {{4}, {{{4, 4}, {10}}}}}}}}},
  {{{1, 1}, {0, {}}}, {{1, 2}, {0, {}}}, {{1, 3}, {0, {}}}, {{1, 4}, {0, {}}},
   {{2, 2}, {0, {}}}, {{2, 3}, {0, {}}}, {{2, 4}, {0, {}}},
   {{3, 3}, {0, {}}}, {{3, 4}, {1, {}}}, {{4, 4}, {0, {}}}}}}
```

```
meanMeanStep[{fructifiedFateTree[[1]], %}, strategy1[[1]], withStrategy → True]
```

```
{0,
 {{{{}, {1/32, {{{1, 1}, {{{1}, {2}}}}, {{1, 2}, {{{1, 2}, {7}}}}, {{1, 3}, {{{1}, {2}}}}, {{1, 4},
      {{{1}, {2}}}}, {{2, 2}, {{{2}, {3}}}}, {{2, 3}, {{{2}, {3}}}}, {{2, 4}, {{{2}, {3}}}},
     {{3, 3}, {{{}, {1}}}}, {{3, 4}, {{{}, {1}}}}, {{4, 4}, {{{}, {1}}}}}}}, {}}}
```

Check: when aiming at {1, 2}, reaching actually {3, 4} (a complete failure) implies raising at last cast {3, 4} after obtaining at first cast {3, 4} (prob. 1/8) or {3, 3} (prob. 1/16) or {4, 4}.

$$\frac{1}{8}\,\frac{1}{8} + 2\,\frac{1}{16}\,\frac{1}{8} == \frac{1}{32}$$

```
True
```

```
{fructifiedFateTree, strategy1} =.;
```

⚠ Remember section 5.2.1: shared state positions in fate tree and strategy must be consistent.

```
Clear@utilityAndStrategy;
utilityAndStrategy[fateTree1_, utility_, options___]@strategy1_List :=
 With[{fructifiedFateTree = fructify[utility]@fateTree1},
  reverseFoldList[meanMeanStep[{First@#1, #2}, Last@#1, options] &,
   Transpose[dropLast /@ {fructifiedFateTree, strategy1}], Last@fructifiedFateTree]
 ]
```

Check strategy echo.

```
With[{d = 2, f = 2, j = 3},
 With[{fateTree1 = fateTree["p1", d, f, j]},
  With[{strategy1 = extractStrategy@
      maxUtilityAndStrategy[fateTree1, δ[Last@allRepresentatives[d, f], #2] &]},
   Equal @@ {extractStrategy@utilityAndStrategy[fateTree1, -10 #1 + Total@#2 &,
        withStrategy → True]@strategy1, strategy1}]]]
```

```
True
```

```
With[{d = 2, f = 2, j = 3},
 With[{fateTree1 = fateTree["p1", d, f, j]},
  With[{strategy1 = extractStrategy@
      maxUtilityAndStrategy[fateTree1, δ[Last@allRepresentatives[d, f], #2] &]},
   utilityAndStrategy[fateTree1, -10 #1 + Total@#2 &]@strategy1]]]
```

```
{{0, {{{{}, {-29/2, {__}}}}, {}}},
 {1, {{{{}, {-22, {__}}}, {{1}, {-89/4, {__}}}, {{2}, {-87/4, {__}}}},
   {{{1, 1}, {-8, {}}}, {{1, 2}, {-7, {}}}, {{2, 2}, {-6, {}}}}}},
 {2, {{{{}, {-27, {__}}}, {{1}, {-55/2, {__}}}, {{2}, {-53/2, {__}}}},
   {{{1, 1}, {-18, {}}}, {{1, 2}, {-17, {}}}, {{2, 2}, {-16, {}}}}}},
 {3, {{}, {{{1, 1}, {-28, {}}}, {{1, 2}, {-27, {}}}, {{2, 2}, {-26, {}}}}}}}
```

```
With[{d = 2, f = 2, j = 3},
 With[{fateTree1 = fateTree["p1", d, f, j]},
  initial@utilityAndStrategy[fateTree1, -10 #1 + Total@#2 &]@
    extractStrategy@maxUtilityAndStrategy[fateTree1, δ[Last@allRepresentatives[d, f], #2] &]
 ]
]
```

$-\frac{29}{2}$

5.3. Consistency between mean-max and mean-mean

5.3.1. Checking mean-max utility from strategy, with mean-mean

5.3.2. Mean-max strategy is more useful than other strategies

5.3.3. Checking pure strategy utility with forcing and mean-max

5.4. Displaying strategies with mean-mean and symbolic utility

5.4.1. Displaying one strategy state branch

5.4.2. Displaying last step

5.4.3. Displaying completely

6. Most probably successful constant-goal strategies

6.1. A stratagem of mean-max: the ratchet policy

6.1.1. First player

6.1.2. Next players' dilemmas

6.2. Evaluating probabilities

6.2.1. All-time success probabilities

6.2.2. Cast number probabilities

6.2.3. Formula for brelan constant-goal cast number probabilities

6.3. Compiling final state probabilities

6.3.1. Definition and evaluation

6.3.2. Dynamic self-similarity

6.3.3. Constant final date and state

6.3.4. Variable final date and state

6.3.5. Variable goal with domain restrictions

6.3.6. Saving and retrieving

6.4. Final state probability charts

6.4.1. Success probabilities

All-time success probabilities, meaningful for first player, are also given.

```
{d = 2, f = 3, j = 3}; Block[{probability},
 TableForm[
   With[{goal = Table[1, {d}]},
    {
     {#, smallBox@Total@#} &[probability["p1", f, j, goal, #, goal] & /@ Range@j],
     {probability["p2", f, #, goal, #, goal] & /@ Range@j}
    }
   ],
   TableDirections → {Row, Row, Column},
   TableHeadings → {{"First player", "Next players"}}] /. probability → π]
```

First player		Next players
π[p1, 3, 3, {1, 1}, 1, {1, 1}]	π[p1, 3, 3, {1, 1}, 1, {1, 1}] +	π[p2, 3, 1, {1, 1}, 1, {1,
π[p1, 3, 3, {1, 1}, 2, {1, 1}]	π[p1, 3, 3, {1, 1}, 2, {1, 1}] +	π[p2, 3, 2, {1, 1}, 2, {1,
π[p1, 3, 3, {1, 1}, 3, {1, 1}]	π[p1, 3, 3, {1, 1}, 3, {1, 1}]	π[p2, 3, 3, {1, 1}, 3, {1,

```
TableForm[
 With[{goal = #},
    With[{player = #},
       With[
        {probabilities = probability[player, f, #, goal, #, goal] & /@ Range@j}, Switch[player,
         "p1", {#, Total /@ Rest@history@#} &@probabilities ,
         "p2", {#} &@probabilities]
       ]
      ] & /@ {"p1", "p2"}] & /@ allRepresentatives[d, f],
 TableDepth → 4,
 TableDirections → {Column, Row, Row, Column},
 TableHeadings → {allRepresentatives[d, f], {"First player", "Next players"}}]
```

	First player		Next players
{1, 1}	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{19}{81}$
	$\frac{136}{729}$	$\frac{361}{729}$	$\frac{211}{729}$
{1, 2}	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$
	$\frac{20}{81}$	$\frac{38}{81}$	$\frac{26}{81}$
	$\frac{128}{729}$	$\frac{470}{729}$	$\frac{242}{729}$

```
With[{d1 = First@#, j1 = Last@#},
    TableForm[
     With[{goal = #},
        With[{player = #},
           With[{probabilities = probability[player, f, #, goal, #, goal] & /@ Range@j1},
            Switch[player,
             "p1", {#, Total /@ Rest@history@#} &@probabilities ,
             "p2", {#} &@probabilities]
           ]
          ] & /@ {"p1", "p2"}
       ] & /@ allRepresentatives[d1, f],
     TableDepth → 4,
     TableDirections → {Column, Row, Row, Column},
     TableHeadings → {allRepresentatives[d1, f], {"First player", "Next players"}}]
   ] & /@ finalDynamicParameters[d, j] // scanPrint
```

	First player		Next player
{1}	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
	$\frac{2}{9}$	$\frac{5}{9}$	$\frac{1}{3}$

	First player		Next player
{1, 1}	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{19}{81}$
	$\frac{136}{729}$	$\frac{361}{729}$	$\frac{211}{729}$
{1, 2}	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$
	$\frac{20}{81}$	$\frac{38}{81}$	$\frac{26}{81}$
	$\frac{128}{729}$	$\frac{470}{729}$	$\frac{242}{729}$

```
TableForm[Flatten[
  With[{d1 = First@#, j1 = Last@#},
     With[{goal = #},
        With[{player = #},
           With[{probabilities = probability[player, f, #, goal, #, goal] & /@ Range@j1},
            Switch[player,
             "p1", ReplacePart[Map[numberForm1,
               {#, Total /@ Rest@history@#} &@probabilities, {2}], "", {-1, 1}],
             "p2", Map[numberForm1, {#} &@probabilities, {2}]]
            ]
          ] & /@ {"p1", "p2"}
       ] & /@ allRepresentatives[d1, f]
    ] & /@ finalDynamicParameters[d, j], 1],
 TableDirections → {Column, Row, Row, Column},
 TableHeadings →
  {combinationBox /@ Flatten[With[{d1 = First@#, j1 = Last@#}, allRepresentatives[d1, f]] & /@
      finalDynamicParameters[d, j], 1], {"First player", "Next players"}}
]
```

	First player		Next players
1	numberForm1[$\frac{1}{3}$]		numberForm1[$\frac{1}{3}$]
	numberForm1[$\frac{2}{9}$]	numberForm1[$\frac{5}{9}$]	numberForm1[$\frac{1}{3}$]
1 1	numberForm1[$\frac{1}{9}$]		numberForm1[$\frac{1}{9}$]
	numberForm1[$\frac{16}{81}$]	numberForm1[$\frac{25}{81}$]	numberForm1[$\frac{19}{81}$]
	numberForm1[$\frac{136}{729}$]	numberForm1[$\frac{361}{729}$]	numberForm1[$\frac{211}{729}$]
2 1	numberForm1[$\frac{2}{9}$]		numberForm1[$\frac{2}{9}$]
	numberForm1[$\frac{20}{81}$]	numberForm1[$\frac{38}{81}$]	numberForm1[$\frac{26}{81}$]
	numberForm1[$\frac{128}{729}$]	numberForm1[$\frac{470}{729}$]	numberForm1[$\frac{242}{729}$]

```
{d =., f =., j =.};

Clear@successProbabilityChart;
Protect@numberForm1;
Options@successProbabilityChart = {numberForm1 → Identity};
successProbabilityChart[d_Integer, f_Integer, j_Integer, probability, options___] :=
 (Print@"Success probabilities"; Print@TableForm[Flatten[With[{d1 = First@#, j1 = Last@#},
        With[{goal = #},
           With[{player = #},
              With[{probabilities = probability[player, f, #, goal, #, goal] & /@ Range@j1},
                 Switch[player,
                  "p1", ReplacePart[Map[numberForm1,
                     {#, Total /@ Rest@history@#} &@probabilities, {2}], "", {-1, 1}],
                  "p2", Map[numberForm1, {#} &@probabilities, {2}]]
                 ] /. {options} /. Options@successProbabilityChart
              ] & /@ {"p1", "p2"}
           ] & /@ allRepresentatives[d1, f]
        ] & /@ finalDynamicParameters[d, j], 1],
     TableDirections → {Column, Row, Row, Column},
     TableHeadings →
      {combinationBox /@ Flatten[With[{d1 = First@#, j1 = Last@#}, allRepresentatives[d1, f]
           ] & /@ Append[{#, j - 1} & /@ Range[d - 1], {d, j}], 1], {"First player", "Next players"}}
   ])
```

```
successProbabilityChart[2, 3, 3, probability]
```

Success probabilities

	First player		Next players
1	$\frac{1}{3}$		$\frac{1}{3}$
	$\frac{2}{9}$	$\frac{5}{9}$	$\frac{1}{3}$
1 1	$\frac{1}{9}$		$\frac{1}{9}$
	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{19}{81}$
	$\frac{136}{729}$	$\frac{361}{729}$	$\frac{211}{729}$
2 1	$\frac{2}{9}$		$\frac{2}{9}$
	$\frac{20}{81}$	$\frac{38}{81}$	$\frac{26}{81}$
	$\frac{128}{729}$	$\frac{470}{729}$	$\frac{242}{729}$

```
successProbabilityChart[3, 6, 3, probability, numberForm1 → approximationBox]
```

Success probabilities

	First player		Next players
1	$\frac{1}{6} \simeq 0.167$		$\frac{1}{6} \simeq 0.167$
	$\frac{5}{36} \simeq 0.139$	$\frac{11}{36} \simeq 0.306$	$\frac{1}{6} \simeq 0.167$
1 1	$\frac{1}{36} \simeq 0.028$		$\frac{1}{36} \simeq 0.028$
	$\frac{85}{1296} \simeq 0.066$	$\frac{121}{1296} \simeq 0.093$	$\frac{91}{1296} \simeq 0.070$
2 1	$\frac{1}{18} \simeq 0.056$		$\frac{1}{18} \simeq 0.056$
	$\frac{35}{324} \simeq 0.108$	$\frac{53}{324} \simeq 0.164$	$\frac{19}{162} \simeq 0.117$
1 1 1	$\frac{1}{216} \simeq 0.005$		$\frac{1}{216} \simeq 0.005$
	$\frac{1115}{46656} \simeq 0.024$	$\frac{1331}{46656} \simeq 0.029$	$\frac{1151}{46656} \simeq 0.025$
	$\frac{466075}{10077696} \simeq 0.046$	$\frac{753571}{10077696} \simeq 0.075$	$\frac{513991}{10077696} \simeq 0.051$
2 1 1	$\frac{1}{72} \simeq 0.014$		$\frac{1}{72} \simeq 0.014$
	$\frac{143}{2592} \simeq 0.055$	$\frac{179}{2592} \simeq 0.069$	$\frac{149}{2592} \simeq 0.057$
	$\frac{23681}{279936} \simeq 0.085$	$\frac{43013}{279936} \simeq 0.154$	$\frac{26903}{279936} \simeq 0.096$
3 2 1	$\frac{1}{36} \simeq 0.028$		$\frac{1}{36} \simeq 0.028$
	$\frac{227}{2592} \simeq 0.088$	$\frac{299}{2592} \simeq 0.115$	$\frac{239}{2592} \simeq 0.092$
	$\frac{21043}{186624} \simeq 0.113$	$\frac{42571}{186624} \simeq 0.228$	$\frac{24631}{186624} \simeq 0.132$

6.4.2. Failure probabilities

Gross classing, depending on the representatives of the birepresentative components.

```
{d = 2, f = 3}; Sort@DeleteCases[allBirepresentatives[d, f], {x_, x_}]
```

```
{{{1, 1}, {1, 2}}, {{1, 1}, {2, 2}}, {{1, 1}, {2, 3}},
 {{1, 2}, {1, 1}}, {{1, 2}, {1, 3}}, {{1, 2}, {3, 3}}}
```

```
listQuotient[
 Sort@DeleteCases[allBirepresentatives[d, f], {x_, x_}],
 {First@#, representative@Last@#} &, withValues → True]
```

```
{{{{1, 1}, {1, 1}}, {{{1, 1}, {2, 2}}}}, {{{1, 1}, {1, 2}}, {{{1, 1}, {1, 2}}, {{1, 1}, {2, 3}}}},
 {{{1, 2}, {1, 1}}, {{{1, 2}, {1, 1}}, {{1, 2}, {3, 3}}}}, {{{1, 2}, {1, 2}}, {{{1, 2}, {1, 3}}}}}
```

```
MapAt[{#, function1 @@ #} & /@ # &, #, -1] & /@ listQuotient[
  Sort@DeleteCases[allBirepresentatives[d, f], {x_, x_}],
  {First@#, representative@Last@#} &, withValues → True]
```

```
{{{{1, 1}, {1, 1}}, {{{{1, 1}, {2, 2}}, function1[{1, 1}, {2, 2}]}}},
 {{{1, 1}, {1, 2}}, {{{{1, 1}, {1, 2}}, function1[{1, 1}, {1, 2}]},
   {{{1, 1}, {2, 3}}, function1[{1, 1}, {2, 3}]}}}, {{{1, 2}, {1, 1}},
  {{{{1, 2}, {1, 1}}, function1[{1, 2}, {1, 1}]}, {{{1, 2}, {3, 3}}, function1[{1, 2}, {3, 3}]}}},
 {{{1, 2}, {1, 2}}, {{{{1, 2}, {1, 3}}, function1[{1, 2}, {1, 3}]}}}}
```

```
fill[MapAt[{#, function1 @@ #} & /@ # &, #, -1] & /@ listQuotient[
   Sort@DeleteCases[allBirepresentatives[d, f], {x_, x_}],
   {First@#, representative@Last@#} &, withValues → True]]
```

```
{{{{{{{1, 1}, {2, 2}}, function1[{1, 1}, {2, 2}]}},
   {{{{1, 1}, {1, 2}}, function1[{1, 1}, {1, 2}]}, {{{1, 1}, {2, 3}}, function1[{1, 1}, {2, 3}]}}},
  {{{{{1, 2}, {1, 1}}, function1[{1, 2}, {1, 1}]}, {{{1, 2}, {3, 3}}, function1[{1, 2}, {3, 3}]}},
   {{{{1, 2}, {1, 3}}, function1[{1, 2}, {1, 3}]}}}}, {{{1, 1}, {1, 2}}, {{1, 1}, {1, 2}}}}
```

```
TableForm[First@#, TableDepth → 4, TableHeadings → Last@#] &@
  fill[MapAt[{#, function1 @@ #} & /@ # &, #, -1] & /@ listQuotient[
     Sort@DeleteCases[allBirepresentatives[d, f], {x_, x_}],
     {First@#, representative@Last@#} &, withValues → True]] /. function1 → "f1"
```

	{1, 1}	{1, 2}
{1, 1}	{{1, 1}, {2, 2}} f1[{1, 1}, {2, 2}]	{{1, 1}, {1, 2}} f1[{1, 1}, {1, 2}] {{1, 1}, {2, 3}} f1[{1, 1}, {2, 3}]
{1, 2}	{{1, 2}, {1, 1}} f1[{1, 2}, {1, 1}] {{1, 2}, {3, 3}} f1[{1, 2}, {3, 3}]	{{1, 2}, {1, 3}} f1[{1, 2}, {1, 3}]

Slice the table so that each piece holds in one page.

```
scanPrint[
 tableFormSlice[1 & /@ allRepresentatives[d, f]][First@#, TableDepth → 4,
    TableHeadings → Last@#] &@fill[MapAt[{#, function1 @@ #} & /@ # &, #, -1] & /@ listQuotient[
     Sort@DeleteCases[allBirepresentatives[d, f], {x_, x_}],
     {First@#, representative@Last@#} &, withValues → True]] /. function1 → "f1"]
```

	{1, 1}	{1, 2}
{1, 1}	{{1, 1}, {2, 2}} f1[{1, 1}, {2, 2}]	{{1, 1}, {1, 2}} f1[{1, 1}, {1, 2}] {{1, 1}, {2, 3}} f1[{1, 1}, {2, 3}]

	{1, 1}	{1, 2}
{1, 2}	{{1, 2}, {1, 1}} f1[{1, 2}, {1, 1}] {{1, 2}, {3, 3}} f1[{1, 2}, {3, 3}]	{{1, 2}, {1, 3}} f1[{1, 2}, {1, 3}]

```
player = "p1"; j = 3;
```

```
scanPrint@With[{d1 = d, j1 = j},
  tableFormSlice[1 & /@ allRepresentatives[d1, f]][First@#,
    TableDepth → 4, TableHeadings → Last@#] &@fill[MapAt[{#,
          With[{goal = First@#, finalState = Last@#},
           probability[player, f, #, goal, #, finalState] & /@ Range@j1
          ]
         } & /@ # &, #, -1] & /@ listQuotient[
     Sort@DeleteCases[allBirepresentatives[d, f], {x_, x_}],
     {First@#, representative@Last@#} &, withValues → True]]
 ]
```

	{1, 1}	{1, 2}
{1, 1}	{{1, 1}, {2, 2}} $\{\frac{1}{9}, \frac{4}{81}, \frac{16}{729}\}$	{{1, 1}, {1, 2}} $\{\frac{2}{9}, \frac{20}{81}, \frac{152}{729}\}$ {{1, 1}, {2, 3}} $\{\frac{2}{9}, \frac{8}{81}, \frac{32}{729}\}$

	{1, 1}	{1, 2}
{1, 2}	{{1, 2}, {1, 1}} $\{\frac{1}{9}, \frac{10}{81}, \frac{64}{729}\}$ {{1, 2}, {3, 3}} $\{\frac{1}{9}, \frac{1}{81}, \frac{1}{729}\}$	{{1, 2}, {1, 3}} $\{\frac{2}{9}, \frac{11}{81}, \frac{65}{729}\}$

`TableForm` bug: the line entry is missing in first table.

```
scanPrint@Flatten[
  With[{d1 = First@#, j1 = Last@#},
     tableFormSlice[1 & /@ allRepresentatives[d1, f]][First@#,
        TableHeadings → Last@#,
        TableSpacing → {1, 1, 3, 1},
        TableAlignments → {Center, Center, Center, Center}
       ] &@fill[MapAt[{
              ColumnForm[Insert[#, "↓", 2], Center],
              With[{goal = First@#, finalState = Last@#},
               TableForm[
                Map[numberForm1, probability[player, f, #, goal, #, finalState] & /@ Range@j1]
               ]
              ]
             } & /@ # &, #, -1] & /@ listQuotient[
         Sort@DeleteCases[allBirepresentatives[d1, f], {x_, x_}],
         {First@#, representative[Last@#]} &, withValues → True]]
    ] & /@ finalDynamicParameters[d, j],
  1]
```

```
                     {1}
{1}  numberForm1[1/3]
 ↓
{2}  numberForm1[2/9]

                 {1, 1}                          {1, 2}
                                      {1, 1}  numberForm1[2/9]
                                        ↓     numberForm1[20/81]
                                      {1, 2}  numberForm1[152/729]
        {1, 1}  numberForm1[1/9]
{1, 1}    ↓     numberForm1[4/81]
        {2, 2}  numberForm1[16/729]   {1, 1}  numberForm1[2/9]
                                        ↓     numberForm1[8/81]
                                      {2, 3}  numberForm1[32/729]

                 {1, 1}                          {1, 2}
        {1, 2}  numberForm1[1/9]
          ↓     numberForm1[10/81]
        {1, 1}  numberForm1[64/729]   {1, 2}  numberForm1[2/9]
{1, 2}                                  ↓     numberForm1[11/81]
        {1, 2}  numberForm1[1/9]      {1, 3}  numberForm1[65/729]
          ↓     numberForm1[1/81]
        {3, 3}  numberForm1[1/729]
```

Patch.

```
scanPrint@MapAt[TableForm[{{1}, #}, TableDirections → Row] &, Flatten[
   With[{d1 = First@#, j1 = Last@#},
      tableFormSlice[1 & /@ allRepresentatives[d1, f]][First@#,
         TableHeadings → Last@#,
         TableSpacing → {1, 1, 3, 1},
         TableAlignments → {Center, Center, Center, Center}
        ] &@fill[MapAt[{
               ColumnForm[Insert[#, "↓", 2], Center],
               With[{goal = First@#, finalState = Last@#},
                TableForm[
                 Map[numberForm1, probability[player, f, #, goal, #, finalState] & /@ Range@j1]
                ]
               ]
              } & /@ # &, #, -1] & /@ listQuotient[
          Sort@DeleteCases[allBirepresentatives[d1, f], {x_, x_}],
          {First@#, representative@Last@#} &, withValues → True]]
     ] & /@ finalDynamicParameters[d, j], 1], 1]
```

```
                              {1}
1    {1}    numberForm1[1/3]
      ↓
     {2}    numberForm1[2/9]

                    {1, 1}                                  {1, 2}
                                            {1, 1}   numberForm1[2/9]
                                              ↓      numberForm1[20/81]
        {1, 1}   numberForm1[1/9]           {1, 2}   numberForm1[152/729]
{1, 1}    ↓      numberForm1[4/81]
        {2, 2}   numberForm1[16/729]        {1, 1}   numberForm1[2/9]
                                              ↓      numberForm1[8/81]
                                            {2, 3}   numberForm1[32/729]

                    {1, 1}                                  {1, 2}
        {1, 2}   numberForm1[1/9]
          ↓      numberForm1[10/81]
        {1, 1}   numberForm1[64/729]        {1, 2}   numberForm1[2/9]
{1, 2}                                        ↓      numberForm1[11/81]
        {1, 2}   numberForm1[1/9]           {1, 3}   numberForm1[65/729]
          ↓      numberForm1[1/81]
        {3, 3}   numberForm1[1/729]
```

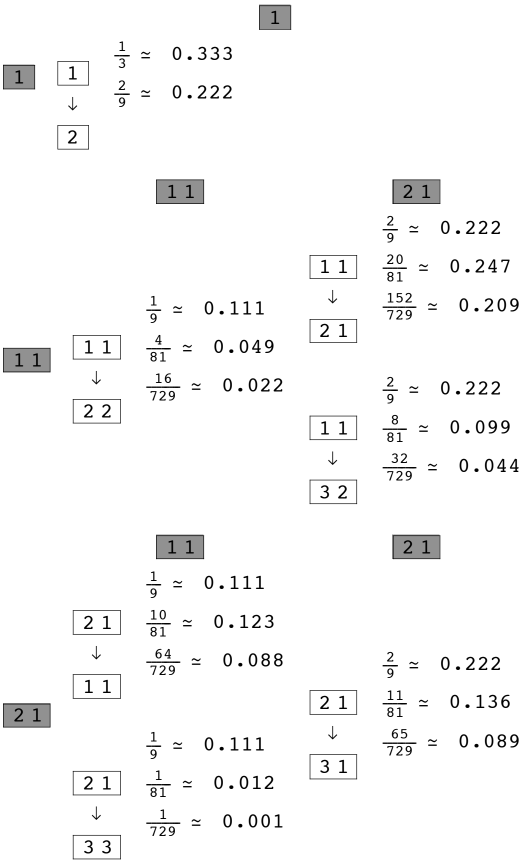
```
{player, d, f, j} =.;
```

When **TableForm** bug is repaired, replace patch by identity.

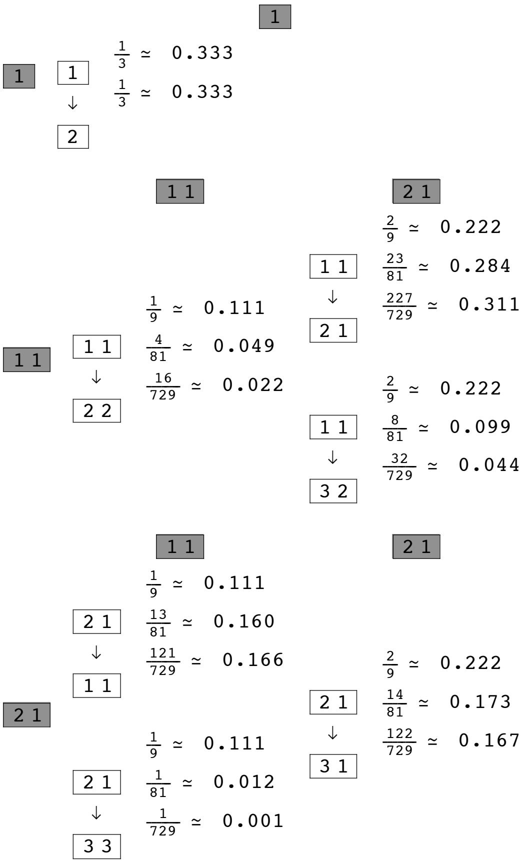
```
Clear@failureProbabilityChart;
Options@failureProbabilityChart = {numberForm1 → Identity};
failureProbabilityChart[player : "p1" | "p2",
  d_Integer, f_Integer, j_Integer, probability, options___] :=
 With[{patch = MapAt[TableForm[{grayCombinationBox@{1}, #}, TableDirections → Row] &, #, 1] &},
  Switch[player,
   "p1", Print["First player's failure probabilities"],
   "p2", Print["Next players' failure probabilities"]];
  scanPrint@patch@Flatten[
     With[{d1 = First@#, j1 = Last@#},
        tableFormSlice[1 & /@ allRepresentatives[d1, f]][First@#,
           TableHeadings → Map[grayCombinationBox, Last@#, {2}],
           TableSpacing → {1, 1, 3, 1}
          ] &@fill[MapAt[{
                 ColumnForm[Insert[combinationBox /@ #, "↓", 2], Center],
                 With[{goal = First@#, finalState = Last@#},
                  TableForm[
                   With[{probabilities =
                       probability[player, f, #, goal, #, finalState] & /@ Range@j1},
                    Map[numberForm1 /. {options} /. Options@failureProbabilityChart,
                     probabilities]
                   ]
                  ]
                 ]
                } & /@ # &, #, -1] & /@ listQuotient[
            Sort@DeleteCases[allBirepresentatives[d1, f], {x_, x_}],
            {First@#, representative@Last@#} &, withValues → True]]
       ] & /@ finalDynamicParameters[d, j], 1]]
```

```
With[{d = 2, f = 3, j = 3},
 failureProbabilityChart["p1", d, f, j, probability, numberForm1 → approximationBox];
 failureProbabilityChart["p2", d, f, j, probability, numberForm1 → approximationBox]
]
```

First player's failure probabilities



Next players' failure probabilities



```
With[{d = 3, f = 6, j = 3},
failureProbabilityChart["p1", d, f, j, probability, numberForm1 → approximationBox];
 failureProbabilityChart["p2", d, f, j, probability, numberForm1 → approximationBox]
]
```

First player's failure probabilities

	1
1	1 → 2: $\frac{1}{6} \simeq 0.167$, $\frac{5}{36} \simeq 0.139$

	1 1	2 1
1 1	1 1 → 2 2: $\frac{1}{36} \simeq 0.028$, $\frac{25}{1296} \simeq 0.019$	1 1 → 2 1: $\frac{1}{18} \simeq 0.056$, $\frac{55}{648} \simeq 0.085$ 1 1 → 3 2: $\frac{1}{18} \simeq 0.056$, $\frac{25}{648} \simeq 0.039$

	1 1	2 1
2 1	2 1 → 1 1: $\frac{1}{36} \simeq 0.028$, $\frac{35}{648} \simeq 0.054$ 2 1 → 3 3: $\frac{1}{36} \simeq 0.028$, $\frac{1}{81} \simeq 0.012$	2 1 → 3 1: $\frac{1}{18} \simeq 0.056$, $\frac{43}{648} \simeq 0.066$ 2 1 → 4 3: $\frac{1}{18} \simeq 0.056$, $\frac{2}{81} \simeq 0.025$

	1 1 1	2 1 1	3 2 1
1 1 1	1 1 1 → 2 2 2: $\frac{1}{216} \simeq 0.005$, $\frac{125}{46656} \simeq 0.003$, $\frac{15625}{10077696} \simeq 0.002$	1 1 1 → 2 1 1: $\frac{1}{72} \simeq 0.014$, $\frac{605}{15552} \simeq 0.039$, $\frac{207025}{3359232} \simeq 0.062$ 1 1 1 → 2 2 1: $\frac{1}{72} \simeq 0.014$, $\frac{275}{15552} \simeq 0.018$, $\frac{56875}{3359232} \simeq 0.017$ 1 1 1 → 3 2 2: $\frac{1}{72} \simeq 0.014$, $\frac{125}{15552} \simeq 0.008$, $\frac{15625}{3359232} \simeq 0.005$	1 1 1 → 3 2 1: $\frac{1}{36} \simeq 0.028$, $\frac{275}{7776} \simeq 0.035$, $\frac{56875}{1679616} \simeq 0.034$ 1 1 1 → 4 3 2: $\frac{1}{36} \simeq 0.028$, $\frac{125}{7776} \simeq 0.016$, $\frac{15625}{1679616} \simeq 0.009$

1 1 1 | 2 1 1 | 3 2 1

2 1 1

2 1 1 → 3 1 1: $\frac{1}{72} \simeq 0.014$, $\frac{103}{3888} \simeq 0.026$, $\frac{6499}{209952} \simeq 0.031$

2 1 1 → 1 1 1: $\frac{1}{216} \simeq 0.005$, $\frac{205}{11664} \simeq 0.018$, $\frac{16105}{629856} \simeq 0.026$

2 1 1 → 2 2 1: $\frac{1}{72} \simeq 0.014$, $\frac{91}{1944} \simeq 0.047$, $\frac{28219}{419904} \simeq 0.067$

2 1 1 → 3 2 1: $\frac{1}{36} \simeq 0.028$, $\frac{37}{648} \simeq 0.057$, $\frac{10217}{139968} \simeq 0.073$

2 1 1 → 2 2 2: $\frac{1}{216} \simeq 0.005$, $\frac{215}{23328} \simeq 0.009$, $\frac{20605}{2519424} \simeq 0.008$

2 1 1 → 3 3 1: $\frac{1}{72} \simeq 0.014$, $\frac{5}{486} \simeq 0.010$, $\frac{38}{6561} \simeq 0.006$

2 1 1 → 4 3 1: $\frac{1}{36} \simeq 0.028$, $\frac{5}{243} \simeq 0.021$, $\frac{76}{6561} \simeq 0.012$

2 1 1 → 3 3 3: $\frac{1}{216} \simeq 0.005$, $\frac{1}{729} \simeq 0.001$, $\frac{8}{19683} \simeq 4.\times 10^{-4}$

2 1 1 → 3 2 2: $\frac{1}{72} \simeq 0.014$, $\frac{77}{3888} \simeq 0.020$, $\frac{7039}{419904} \simeq 0.017$

2 1 1 → 4 3 2: $\frac{1}{36} \simeq 0.028$, $\frac{31}{1296} \simeq 0.024$, $\frac{839}{46656} \simeq 0.018$

2 1 1 → 3 3 2: $\frac{1}{72} \simeq 0.014$, $\frac{31}{2592} \simeq 0.012$, $\frac{839}{93312} \simeq 0.009$

2 1 1 → 5 4 3: $\frac{1}{36} \simeq 0.028$, $\frac{2}{243} \simeq 0.008$, $\frac{16}{6561} \simeq 0.002$

2 1 1 → 4 3 3: $\frac{1}{72} \simeq 0.014$, $\frac{1}{243} \simeq 0.004$, $\frac{8}{6561} \simeq 0.001$

1 1 1 | 2 1 1 | 3 2 1

3 2 1

3 2 1 → 2 1 1: $\frac{1}{72} \simeq 0.014$, $\frac{179}{5184} \simeq 0.035$, $\frac{15067}{373248} \simeq 0.040$

3 2 1 → 4 2 1: $\frac{1}{36} \simeq 0.028$, $\frac{319}{7776} \simeq 0.041$, $\frac{24239}{559872} \simeq 0.043$

3 2 1 → 1 1 1: $\frac{1}{216} \simeq 0.005$, $\frac{83}{15552} \simeq 0.005$, $\frac{3115}{1119744} \simeq 0.003$

3 2 1 → 4 1 1: $\frac{1}{72} \simeq 0.014$, $\frac{175}{15552} \simeq 0.011$, $\frac{6311}{1119744} \simeq 0.006$

3 2 1 → 5 4 1: $\frac{1}{36} \simeq 0.028$, $\frac{101}{7776} \simeq 0.013$, $\frac{3277}{559872} \simeq 0.006$

3 2 1 → 4 4 4: $\frac{1}{216} \simeq 0.005$, $\frac{1}{1728} \simeq 6.\times 10^{-4}$, $\frac{1}{13824} \simeq 7.\times 10^{-5}$

3 2 1 → 4 4 1: $\frac{1}{72} \simeq 0.014$, $\frac{101}{15552} \simeq 0.006$, $\frac{3277}{1119744} \simeq 0.003$

3 2 1 → 6 5 4: $\frac{1}{36} \simeq 0.028$, $\frac{1}{288} \simeq 0.003$, $\frac{1}{2304} \simeq 4.\times 10^{-4}$

3 2 1 → 5 4 4: $\frac{1}{72} \simeq 0.014$, $\frac{1}{576} \simeq 0.002$, $\frac{1}{4608} \simeq 2.\times 10^{-4}$

Next players' failure probabilities

1

1

1 → 2: $\frac{1}{6} \simeq 0.167$, $\frac{1}{6} \simeq 0.167$

1 1	1 1	1 1 $\downarrow$ 2 2	$\frac{1}{36} \simeq 0.028$; $\frac{25}{1296} \simeq 0.019$
1 1	2 1	1 1 $\downarrow$ 2 1	$\frac{1}{18} \simeq 0.056$; $\frac{29}{324} \simeq 0.090$
1 1	2 1	1 1 $\downarrow$ 3 2	$\frac{1}{18} \simeq 0.056$; $\frac{25}{648} \simeq 0.039$

2 1	1 1	2 1 $\downarrow$ 1 1	$\frac{1}{36} \simeq 0.028$; $\frac{19}{324} \simeq 0.059$
2 1	1 1	2 1 $\downarrow$ 3 3	$\frac{1}{36} \simeq 0.028$; $\frac{1}{81} \simeq 0.012$
2 1	2 1	2 1 $\downarrow$ 3 1	$\frac{1}{18} \simeq 0.056$; $\frac{23}{324} \simeq 0.071$
2 1	2 1	2 1 $\downarrow$ 4 3	$\frac{1}{18} \simeq 0.056$; $\frac{2}{81} \simeq 0.025$

1 1 1	1 1 1	1 1 1 $\downarrow$ 2 2 2	$\frac{1}{216} \simeq 0.005$; $\frac{125}{46656} \simeq 0.003$; $\frac{15625}{10077696} \simeq 0.002$
1 1 1	2 1 1	1 1 1 $\downarrow$ 2 1 1	$\frac{1}{72} \simeq 0.014$; $\frac{617}{15552} \simeq 0.040$; $\frac{222997}{3359232} \simeq 0.066$
1 1 1	2 1 1	1 1 1 $\downarrow$ 2 2 1	$\frac{1}{72} \simeq 0.014$; $\frac{275}{15552} \simeq 0.018$; $\frac{56875}{3359232} \simeq 0.017$
1 1 1	2 1 1	1 1 1 $\downarrow$ 3 2 2	$\frac{1}{72} \simeq 0.014$; $\frac{125}{15552} \simeq 0.008$; $\frac{15625}{3359232} \simeq 0.005$
1 1 1	3 2 1	1 1 1 $\downarrow$ 3 2 1	$\frac{1}{36} \simeq 0.028$; $\frac{275}{7776} \simeq 0.035$; $\frac{56875}{1679616} \simeq 0.034$
1 1 1	3 2 1	1 1 1 $\downarrow$ 4 3 2	$\frac{1}{36} \simeq 0.028$; $\frac{125}{7776} \simeq 0.016$; $\frac{15625}{1679616} \simeq 0.009$

2 1 1	1 1 1	2 1 1 → 1 1 1	$\frac{1}{216} \simeq 0.005$	$\frac{437}{23328} \simeq 0.019$	$\frac{18751}{629856} \simeq 0.030$
2 1 1	1 1 1	2 1 1 → 2 2 2	$\frac{1}{216} \simeq 0.005$	$\frac{215}{23328} \simeq 0.009$	$\frac{20605}{2519424} \simeq 0.008$
2 1 1	1 1 1	2 1 1 → 3 3 3	$\frac{1}{216} \simeq 0.005$	$\frac{1}{729} \simeq 0.001$	$\frac{8}{19683} \simeq 4. \times 10^{-4}$
2 1 1	2 1 1	2 1 1 → 3 1 1	$\frac{1}{72} \simeq 0.014$	$\frac{215}{7776} \simeq 0.028$	$\frac{7381}{209952} \simeq 0.035$
2 1 1	2 1 1	2 1 1 → 2 2 1	$\frac{1}{72} \simeq 0.014$	$\frac{373}{7776} \simeq 0.048$	$\frac{3911}{52488} \simeq 0.075$
2 1 1	2 1 1	2 1 1 → 3 3 1	$\frac{1}{72} \simeq 0.014$	$\frac{5}{486} \simeq 0.010$	$\frac{38}{6561} \simeq 0.006$
2 1 1	2 1 1	2 1 1 → 3 2 2	$\frac{1}{72} \simeq 0.014$	$\frac{77}{3888} \simeq 0.020$	$\frac{7039}{419904} \simeq 0.017$
2 1 1	2 1 1	2 1 1 → 3 3 2	$\frac{1}{72} \simeq 0.014$	$\frac{31}{2592} \simeq 0.012$	$\frac{839}{93312} \simeq 0.009$
2 1 1	2 1 1	2 1 1 → 4 3 3	$\frac{1}{72} \simeq 0.014$	$\frac{1}{243} \simeq 0.004$	$\frac{8}{6561} \simeq 0.001$
2 1 1	3 2 1	2 1 1 → 3 2 1	$\frac{1}{36} \simeq 0.028$	$\frac{151}{2592} \simeq 0.058$	$\frac{1405}{17496} \simeq 0.080$
2 1 1	3 2 1	2 1 1 → 4 3 1	$\frac{1}{36} \simeq 0.028$	$\frac{5}{243} \simeq 0.021$	$\frac{76}{6561} \simeq 0.012$
2 1 1	3 2 1	2 1 1 → 4 3 2	$\frac{1}{36} \simeq 0.028$	$\frac{31}{1296} \simeq 0.024$	$\frac{839}{46656} \simeq 0.018$
2 1 1	3 2 1	2 1 1 → 5 4 3	$\frac{1}{36} \simeq 0.028$	$\frac{2}{243} \simeq 0.008$	$\frac{16}{6561} \simeq 0.002$

3 2 1	1 1 1	3 2 1 → 1 1 1	$\frac{1}{216} \simeq 0.005$	$\frac{83}{15552} \simeq 0.005$	$\frac{3115}{1119744} \simeq 0.003$
3 2 1	1 1 1	3 2 1 → 4 4 4	$\frac{1}{216} \simeq 0.005$	$\frac{1}{1728} \simeq 6. \times 10^{-4}$	$\frac{1}{13824} \simeq 7. \times 10^{-5}$
3 2 1	2 1 1	3 2 1 → 2 1 1	$\frac{1}{72} \simeq 0.014$	$\frac{187}{5184} \simeq 0.036$	$\frac{17459}{373248} \simeq 0.047$
3 2 1	2 1 1	3 2 1 → 4 1 1	$\frac{1}{72} \simeq 0.014$	$\frac{175}{15552} \simeq 0.011$	$\frac{6311}{1119744} \simeq 0.006$
3 2 1	2 1 1	3 2 1 → 4 4 1	$\frac{1}{72} \simeq 0.014$	$\frac{101}{15552} \simeq 0.006$	$\frac{3277}{1119744} \simeq 0.003$
3 2 1	2 1 1	3 2 1 → 5 4 4	$\frac{1}{72} \simeq 0.014$	$\frac{1}{576} \simeq 0.002$	$\frac{1}{4608} \simeq 2. \times 10^{-4}$
3 2 1	3 2 1	3 2 1 → 4 2 1	$\frac{1}{36} \simeq 0.028$	$\frac{331}{7776} \simeq 0.043$	$\frac{27827}{559872} \simeq 0.050$
3 2 1	3 2 1	3 2 1 → 5 4 1	$\frac{1}{36} \simeq 0.028$	$\frac{101}{7776} \simeq 0.013$	$\frac{3277}{559872} \simeq 0.006$
3 2 1	3 2 1	3 2 1 → 6 5 4	$\frac{1}{36} \simeq 0.028$	$\frac{1}{288} \simeq 0.003$	$\frac{1}{2304} \simeq 4. \times 10^{-4}$

6.5. Consistency of probabilities

6.5.1. Dynamic domain

6.5.2. Invariance modulo face permutations

6.5.3. Symmetry breaking due to purifying

6.5.4. Unit probability sum over all final dates and states

6.5.5. All-time success probabilities as time sums

6.5.6. Inequalities on success probabilities

6.5.7. First player success probabilities do not depend on greatest cast number

6.5.8. Failure probabilities are the same for all players if goal and final state differ by at least 2 dice

6.5.9. One-cast probabilities

6.5.10. Formula for complete failure probabilites ($f = 2\,d$)

6.6. Statistics (Monte Carlo)

6.6.1. Simulating ratchet policy

6.6.2. All-time success statistics

6.6.3. Comparison with probabilities

7. Goal-driven policies and goal-driven strategy judging

7.1. Goal-driven utility functions

7.1.1. Relative goal utility, with or without serendipity

```
Clear@goalUtility;

goalUtility[_, _, _, j1_, state1_List, utility_, _, _]@emptyGoal : {} := utility[j1, state1]

goalUtility[player_, f_, j_, j1_, state1_List, utility_, probability, serendipity_]@goal_List :=
 Switch[player,
   "p1", Sum[probability[player, f, j - j1, goal, j2 - j1, goal]
     utility[j2, Sort@Join[state1, goal]], {j2, j1 + 1, j - 1}],
   _, 0] + Total[probability[player, f, j - j1, goal, j - j1, #] utility[j, Sort@Join[state1, #]]
    & /@ If[serendipity, allCombinations[Length@goal, f], {goal}]]

{d = 2, f = 3, j =.}; utility =.;
goalUtility["p1", f, j, 1, {1, 1}, utility, probability, True]@{}

utility[1, {1, 1}]

Table[f, {d - 1}]
goalUtility["p2", f, j, 1, {1}, utility, probability, True]@%

{3}

probability[p2, 3, -1 + j, {3}, -1 + j, {1}] utility[j, {1, 1}] +
 probability[p2, 3, -1 + j, {3}, -1 + j, {2}] utility[j, {1, 2}] +
 probability[p2, 3, -1 + j, {3}, -1 + j, {3}] utility[j, {1, 3}]

goalUtility["p1", f, j, 1, {f}, utility, probability, True]@Table[f, {d - 1}]
```

$$\sum_{j2=2}^{-1+j} \text{probability}[p1, 3, -1+j, \{3\}, -1+j2, \{3\}]\ \text{utility}[j2, \{3, 3\}] +$$

```
 probability[p1, 3, -1 + j, {3}, -1 + j, {1}] utility[j, {1, 3}] +
 probability[p1, 3, -1 + j, {3}, -1 + j, {2}] utility[j, {2, 3}] +
 probability[p1, 3, -1 + j, {3}, -1 + j, {3}] utility[j, {3, 3}]
```

```
goalUtility["p1", f, j, 0, {}, utility, probability, False]@Range@d
```

$$\sum_{j2=1}^{-1+j} \text{probability}[p1, 3, j, \{1, 2\}, j2, \{1, 2\}]\ \text{utility}[j2, \{1, 2\}] + \text{probability}[p1, 3, j, \{1, 2\}, j, \{1, 2\}]\ \text{utility}[j, \{1, 2\}]$$

```
j = 3; utility = only@Cases[utilityList[d, f, j], transfer[__]@_ &]
```

```
transfer[2, 3][#2] &
```

```
allCombinations[d, f]
utility[_, #] & /@ %
```

```
{{1, 1}, {1, 2}, {1, 3}, {2, 2}, {2, 3}, {3, 3}}
```

```
{4, 6, 3, 2, 1, 3}
```

```
goalUtility["p2", f, j, 0, {}, utility, probability, False] /@ allCombinations[d, f]
1. %
```

$$\left\{\frac{844}{729}, \frac{484}{243}, \frac{242}{243}, \frac{422}{729}, \frac{242}{729}, \frac{211}{243}\right\}$$

```
{1.15775, 1.99177, 0.995885, 0.578875, 0.331962, 0.868313}
```

Goal utility depends on serendipity.

```
Ordering[goalUtility["p2", f, j, 0, {}, utility, probability, False] /@ allCombinations[d, f]]
Ordering[goalUtility["p2", f, j, 0, {}, utility, probability, True] /@ allCombinations[d, f]]
```

```
{5, 4, 6, 3, 1, 2}
```

```
{6, 5, 4, 3, 2, 1}
```

Check: for a stationary utility function, the utility of a goal without serendipity is the all-time success probability multiplied by the utility of reaching the goal.

```
utility = utility1@#2 &;
And @@ Flatten[
  With[{player = #},
     With[{d1 = First@#, j1 = Last@#}, With[{fateTree1 = fateTree[player, d1, f, j1]},
        With[{goal = #},
           anyTimeSuccessProbability[fateTree1, goal] utility[_, goal] ==
            goalUtility[player, f, j1, 0, {}, utility, probability, False]@goal
          ] & /@ allRepresentatives[d1, f]]
       ] & /@ dynamicParameters[d, j]
    ] & /@ {"p1", "p2"}]
```

```
True
```

Check: with serendipity, goal utility also is the constant-goal strategy utility (for the same goal).

```
utility =.;
And @@ Simplify /@ Flatten[
   With[{player = #},
      With[{d1 = First@#, j1 = Last@#},
         With[{fateTree1 = fateTree[player, d1, f, j1]}, With[{goal = #},
             goalUtility[player, f, j1, 0, {}, utility, probability, True]@goal ==
              initial@utilityAndStrategy[fateTree1, utility]@
                extractStrategy@maxUtilityAndStrategy[fateTree1, δ[goal, #2] &]
            ] & /@ allRepresentatives[d1, f]]
        ] & /@ dynamicParameters[d, j]
     ] & /@ {"p1", "p2"}]
```

```
True
```

⚠ Once probabilities are known, constant-goal strategies can be judged with serendipitous goal utility function (relying on compiled probabilities) instead of and much faster than `mean-mean`.

⚠ Dynamic check.

```
And @@ With[{player = "p1"}, With[{fateTree1 = fateTree[player, d, f, j]},
   Simplify@With[{goal = #}, goalUtility[player, f, j, 0, {}, utility, probability, True]@goal ==
        initial@utilityAndStrategy[fateTree1, utility]@extractStrategy@
           maxUtilityAndStrategy[fateTree1, δ[goal, #2] &]] & /@ allCombinations[d, f]]]
```

```
True
```

```
With[{player = "p1", utility = only@Cases[utilityList[d, f, j], transfer[__]@_ &]},
 With[{fateTree1 = fateTree[player, d, f, j]}, zoomTiming[
   With[{goal = #}, goalUtility[player, f, j, 0, {}, utility, probability, True]@goal] & /@
    allCombinations[d, f], With[{goal = #},
      initial@utilityAndStrategy[fateTree1, utility]@extractStrategy@maxUtilityAndStrategy[
          fateTree1, δ[goal, #2] &]] & /@ allCombinations[d, f], 30, compare]]]
```

$\{\{0.08\,\text{Second}, 2.67\,\text{Second}\}, \{\text{same}, \{\frac{2924}{729}, \frac{3467}{729}, \frac{2315}{729}, \frac{1994}{729}, \frac{1379}{729}, \frac{1979}{729}\}\}\}$

```
{d =., f =., j =.};
```

7.1.2. Goal-driven utility

7.1.3. Driving goals

7.2. Restricted horizon policies evaluating non-Markovian strategies

7.3. Some dynamic policies

7.3.1. Definition and restriction

7.3.2. Unit horizon

7.3.3. Null horizon: goal explosion

7.4. Compiling strategy utilities

7.4.1. Variable goal-driven parameters and utility function

7.4.2. Hard cases

7.4.3. Completely random strategy and complete most useful strategy

7.5. Consistency of strategy utilities

7.5.1. Invariance modulo face permutations

7.5.2. Formulae for strategy utilities

8. Conclusion

8.1. Policy utility

8.1.1. Definition

8.1.2. Chart

```
Clear@policyUtilityChart;
policyUtilityChart[player : "p1" | "p2", d_Integer, f_Integer, j_Integer] := With[{parameters = {
     utility → # & /@ utilityList[d, f, j],
     serendipity → # & /@ {False, True},
     horizon → # & /@ {0, 1},
     dynamism → # & /@ {False, True}}},
  Print@Switch[player, "p1", "First player", "p2", "Next players"];
  Print@TableForm[Outer[
      Unevaluated@noRational@
         noSymbol@policyUtility[player, d, f, j, utility, serendipity, horizon, dynamism]
        /. {##} &, ##] & @@ parameters, TableHeadings → MapAt[Composition[smallBox, Last] /@ # &,
      parameters, 1] /. {horizon → "h", dynamism → "dyn"}]]
```

"?" stands for undetermined utilities. ⚠ Misalignment: **`TableForm`** bug.

```
Block[{strategyUtility, randomStrategyUtility, maxStrategyUtility},
 << strategyUtility.txt; << randomStrategyUtility.txt;
 << maxStrategyUtility.txt; With[{d = 2, f = 3, j = 3},
  policyUtilityChart["p1", d, f, j];
  policyUtilityChart["p2", d, f, j]];
 With[{d = 3, f = 6, j = 3},
  policyUtilityChart["p1", d, f, j];
  policyUtilityChart["p2", d, f, j]]]
```

First player

		serendipity → False		serendipity → True	
		dyn → False	dyn → True	dyn → False	dyn → True
δ[{1, 2}, #2] &	h → 0	1	1	1	1
	h → 1	1	1	1	1
δ[{1, 2}, #2] + δ[{2, 3}, #2] &	h → 0	0.705	0.705	0.705	0.705
	h → 1	0.807	0.835	0.857	1
δ[{1, 1}, #2] + δ[{2, 2}, #2] &	h → 0	0.686	0.686	0.686	0.686
	h → 1	0.970	1	0.970	1
δ[{1, 1}, #2] + δ[{1, 2}, #2] &	h → 0	0.836	0.836	0.836	0.836
	h → 1	0.888	0.948	0.914	1
transfer[2, 3][#2] &	h → 0	0.952	0.952	0.952	0.952
	h → 1	0.813	0.702	0.980	1
Max[#2] &	h → 0	0.755	0.755	1	1
	h → 1	0.355	0.315	1	1
Total[#2] &	h → 0	0.656	0.756	1	1
	h → 1	0.256	0.200	1	1
Times @@ (#1 - 1 &) /@ #2 &	h → 0	1	1	1	1
	h → 1	1	1	1	1
Times @@ #2 &	h → 0	1	0.941	1	1
	h → 1	0.764	0.680	1	1
randomUtility[2, 3, 3, 280865]	h → 0	0.439	0.439	0.439	0.439
	h → 1	0.885	0.967	0.900	0.963
randomUtility[2, 3, 3, 28086]	h → 0	0.505	0.437	0.505	0.505
	h → 1	0.647	0.673	0.739	0.782
randomUtility[2, 3, 3, 2808]	h → 0	0.394	0.394	0.394	0.394
	h → 1	0.614	0.742	0.699	0.950
randomUtility[2, 3, 3, 280]	h → 0	0.213	0.213	0.291	0.384
	h → 1	0.505	0.773	0.525	0.817
randomUtility[2, 3, 3, 28]	h → 0	0.230	0.230	0.230	0.230
	h → 1	0.599	0.790	0.670	0.952
randomUtility[2, 3, 3, 2]	h → 0	0.383	0.290	0.451	0.576
	h → 1	0.502	0.510	0.902	1

Next players

Utility	h	serendipity → False, dyn → False	serendipity → False, dyn → True	serendipity → True, dyn → False	serendipity → True, dyn → True
δ[{1, 2}, #2] &	h → 0	1	1	1	1
	h → 1	1	1	1	1
δ[{1, 2}, #2] + δ[{2, 3}, #2] &	h → 0	0.308	0.308	1	1
	h → 1	0	0	1	1
δ[{1, 1}, #2] + δ[{2, 2}, #2] &	h → 0	0.813	0.813	1	1
	h → 1	0.963	1	1	1
δ[{1, 1}, #2] + δ[{1, 2}, #2] &	h → 0	0.615	0.615	1	1
	h → 1	0.615	0.615	1	1
transfer[2, 3][#2] &	h → 0	0.492	0.492	1	1
	h → 1	0.492	0.492	1	1
Max[#2] &	h → 0	0.308	0.308	1	1
	h → 1	0	0	1	1
Total[#2] &	h → 0	1	0.938	1	1
	h → 1	0.754	0.754	1	1
Times @@ (#1 - 1 &) /@ #2 &	h → 0	1	1	1	1
	h → 1	1	1	1	1
Times @@ #2 &	h → 0	1	1	1	1
	h → 1	1	1	1	1
randomUtility[2, 3, 3, 280865]	h → 0	0.871	0.989	0.965	0.965
	h → 1	0.989	0.992	0.989	0.996
randomUtility[2, 3, 3, 28086]	h → 0	0.077	0.077	1	1
	h → 1	0.077	0.075	1	1
randomUtility[2, 3, 3, 2808]	h → 0	0.704	0.704	0.968	0.968
	h → 1	0.704	0.686	0.968	1
randomUtility[2, 3, 3, 280]	h → 0	0.905	0.905	0.905	0.976
	h → 1	0.911	0.913	0.976	1
randomUtility[2, 3, 3, 28]	h → 0	0.181	0.181	1	1
	h → 1	0.122	0.122	1	1
randomUtility[2, 3, 3, 2]	h → 0	0.962	0.953	0.962	0.962
	h → 1	0.823	0.835	0.962	1

First player

Utility	h	serendipity → False, dyn → False	serendipity → False, dyn → True	serendipity → True, dyn → False	serendipity → True, dyn → True
δ[{1, 2, 3}, #2] &	h → 0	1	1	1	1
	h → 1	1	1	1	1
δ[{1, 2, 3}, #2] + δ[{4, 5, 6}, #2] &	h → 0	0.684	0.684	0.684	0.684
	h → 1	0.980	0.980	0.980	0.980
δ[{1, 2, 3}, #2] + δ[{2, 3, 4}, #2] &	h → 0	0.806	0.806	0.806	0.806
	h → 1	0.911	0.980	0.911	0.980
δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] &	h → 0	0.605	0.605	0.605	0.605
	h → 1	0.946	1	0.946	1
δ[{1, 2, 3}, #2] + δ[{3, 4, 5}, #2] + δ[{1, 5, 6}, #2] &	h → 0	0.599	0.599	0.599	0.599
	h → 1	0.946	0.960	0.950	0.974
δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] + δ[{1, 1, 3}, #2] + δ[{2, 2, 3}, #2] &	h → 0	0.587	0.587	0.587	0.587
	h → 1	0.865	1	0.865	1
transfer[3, 6][#2] &	h → 0	0.833	0.833	0.833	0.840
	h → 1	0.840	0.828	0.920	0.993
Max[#2] &	h → 0	0.684	?	1	?
	h → 1	0.050	0.050	1	1
Total[#2] &	h → 0	0.768	0.829	0.857	0.893
	h → 1	0.004	0.004	0.915	0.996
Times @@ (#1 - 1 &) /@ #2 &	h → 0	0.778	0.815	0.778	0.819
	h → 1	0.739	0.706	0.857	0.992
Times @@ #2 &	h → 0	0.798	0.835	0.798	0.838
	h → 1	0.507	0.433	0.871	0.993
randomUtility[3, 6, 3, 280865]	h → 0	-0.095	-0.044	0.277	0.303
	h → 1	0.411	0.507	0.697	0.858
randomUtility[3, 6, 3, 28086]	h → 0	0.336	0.348	0.437	0.461
	h → 1	0.620	0.561	0.877	0.974
randomUtility[3, 6, 3, 2808]	h → 0	0.442	0.440	0.479	0.504
	h → 1	0.484	0.531	0.786	0.965
randomUtility[3, 6, 3, 280]	h → 0	0.395	0.395	0.444	0.452
	h → 1	0.397	0.386	0.742	0.926
randomUtility[3, 6, 3, 28]	h → 0	0.400	0.339	0.400	0.406
	h → 1	0.436	0.465	0.761	0.944
randomUtility[3, 6, 3, 2]	h → 0	0.325	0.325	0.338	0.340
	h → 1	0.520	0.516	0.846	0.938

Next players

		serendipity → False		serendipity → True	
		dyn → False	dyn → True	dyn → False	dyn → True
δ[{1, 2, 3}, #2] &	h → 0	1	1	1	1
	h → 1	1	1	1	1
δ[{1, 2, 3}, #2] + δ[{4, 5, 6}, #2] &	h → 0	0.754	0.754	0.754	0.754
	h → 1	0.951	0.951	0.951	0.951
δ[{1, 2, 3}, #2] + δ[{2, 3, 4}, #2] &	h → 0	0.846	0.846	0.846	0.846
	h → 1	0.792	0.801	0.918	1
δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] &	h → 0	0.636	0.636	0.636	0.642
	h → 1	0.930	1	0.933	1
δ[{1, 2, 3}, #2] + δ[{3, 4, 5}, #2] + δ[{1, 5, 6}, #2] &	h → 0	0.756	0.756	0.821	0.821
	h → 1	0.916	0.928	0.944	1
δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] + δ[{1, 1, 3}, #2] + δ[{2, 2, 3}, #2] &	h → 0	0.646	0.646	0.646	0.699
	h → 1	0.876	0.938	0.937	0.987
transfer[3, 6][#2] &	h → 0	0.730	0.730	0.835	0.881
	h → 1	0.718	0.735	0.936	0.998
Max[#2] &	h → 0	0.610	?	1	1
	h → 1	-0.032	-0.032	1	1
Total[#2] &	h → 0	0.805	0.850	0.912	0.945
	h → 1	0.488	0.487	0.955	1
Times @@ (#1 - 1 &) /@ #2 &	h → 0	0.893	0.931	0.893	0.931
	h → 1	0.867	0.882	0.943	1
Times @@ #2 &	h → 0	0.897	0.930	0.897	0.934
	h → 1	0.872	0.858	0.946	1
randomUtility[3, 6, 3, 280865]	h → 0	-0.101	-0.080	0.724	0.770
	h → 1	0.433	0.433	0.929	0.982
randomUtility[3, 6, 3, 28086]	h → 0	0.891	0.891	0.891	0.898
	h → 1	0.516	0.516	0.945	1
randomUtility[3, 6, 3, 2808]	h → 0	0.379	0.454	0.875	0.887
	h → 1	0.349	0.349	0.935	0.990
randomUtility[3, 6, 3, 280]	h → 0	0.659	0.614	0.743	0.821
	h → 1	0.319	0.319	0.925	0.974
randomUtility[3, 6, 3, 28]	h → 0	0.031	0.031	0.773	0.828
	h → 1	0.369	0.369	0.955	0.984
randomUtility[3, 6, 3, 2]	h → 0	0.250	0.250	0.790	0.827
	h → 1	0.224	0.224	0.914	0.988

8.2. Empiric laws of policy utility

8.2.1. Definition: fuzzy utility function

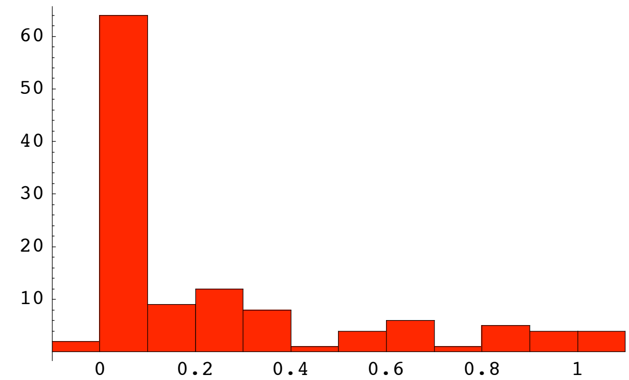
8.2.2. Serendipity is almost always useful (though not strictly)

```
<< Graphics`Graphics`

Clear@serendipityUtilityChart;
serendipityUtilityChart[d_, f_, j_] := With[{serendipityUtilityList =
    Distribute[{{"p1", "p2"}, utilityList[d, f, j], {0, 1}, {False, True}}, List,
     List, List, With[{player = #1, utility = #2, horizon = #3, dynamism = #4},
       {policyUtility[player, d, f, j, utility, True, horizon, dynamism] -
         policyUtility[player, d, f, j, utility, False, horizon, dynamism], player,
        smallBox@utility, horizon, dynamism, noSymbol@fuzzyQ[player, d, f, j, utility]}] &]},
  Print@TableForm[MapAt[Composition[noRational, noSymbol], #, 1] & /@
      Sort@serendipityUtilityList, TableHeadings →
      {None, {"", "", "", horizon, dynamism, fuzzyQ}}, TableSpacing → {1, 1}]; Histogram[
   DeleteCases[noSymbol /@ First /@ serendipityUtilityList, "?"]];]

Block[{strategyUtility, randomStrategyUtility, maxStrategyUtility},
 << strategyUtility.txt; << randomStrategyUtility.txt; << maxStrategyUtility.txt;]

Block[{strategyUtility, randomStrategyUtility, maxStrategyUtility},
 << strategyUtility.txt; << randomStrategyUtility.txt; << maxStrategyUtility.txt;
 serendipityUtilityChart[2, 3, 3]; serendipityUtilityChart[3, 6, 3]]
```

			horizon	dynamism	fuzzyQ
-0.024	p2	randomUtility[2, 3, 3, 280865]	0	True	False
-0.004	p1	randomUtility[2, 3, 3, 280865]	1	True	False
0	p1	Times @@ (#1 - 1 &) /@ #2 &	0	False	False
0	p1	Times @@ (#1 - 1 &) /@ #2 &	0	True	False
0	p1	Times @@ (#1 - 1 &) /@ #2 &	1	False	False
0	p1	Times @@ (#1 - 1 &) /@ #2 &	1	True	False
0	p1	Times @@ #2 &	0	False	False
0	p1	δ[{1, 2}, #2] &	0	False	False
0	p1	δ[{1, 2}, #2] &	0	True	False

0	p1	δ[{1, 2}, #2] &	1	False	False
0	p1	δ[{1, 2}, #2] &	1	True	False
0	p1	δ[{1, 1}, #2] + δ[{1, 2}, #2] &	0	False	False
0	p1	δ[{1, 1}, #2] + δ[{1, 2}, #2] &	0	True	False
0	p1	δ[{1, 1}, #2] + δ[{2, 2}, #2] &	0	False	False
0	p1	δ[{1, 1}, #2] + δ[{2, 2}, #2] &	0	True	False
0	p1	δ[{1, 1}, #2] + δ[{2, 2}, #2] &	1	False	False
0	p1	δ[{1, 1}, #2] + δ[{2, 2}, #2] &	1	True	False
0	p1	δ[{1, 2}, #2] + δ[{2, 3}, #2] &	0	False	False
0	p1	δ[{1, 2}, #2] + δ[{2, 3}, #2] &	0	True	False
0	p1	transfer[2, 3][#2] &	0	False	False
0	p1	transfer[2, 3][#2] &	0	True	False
0	p1	randomUtility[2, 3, 3, 28]	0	False	False
0	p1	randomUtility[2, 3, 3, 28]	0	True	False
0	p1	randomUtility[2, 3, 3, 2808]	0	False	False
0	p1	randomUtility[2, 3, 3, 2808]	0	True	False
0	p1	randomUtility[2, 3, 3, 28086]	0	False	False
0	p1	randomUtility[2, 3, 3, 280865]	0	False	False
0	p1	randomUtility[2, 3, 3, 280865]	0	True	False
0	p2	Times @@ (#1 - 1 &) /@ #2 &	0	False	False
0	p2	Times @@ (#1 - 1 &) /@ #2 &	0	True	False
0	p2	Times @@ (#1 - 1 &) /@ #2 &	1	False	False
0	p2	Times @@ (#1 - 1 &) /@ #2 &	1	True	False
0	p2	Times @@ #2 &	0	False	False
0	p2	Times @@ #2 &	0	True	False
0	p2	Times @@ #2 &	1	False	False
0	p2	Times @@ #2 &	1	True	False
0	p2	δ[{1, 2}, #2] &	0	False	False
0	p2	δ[{1, 2}, #2] &	0	True	False
0	p2	δ[{1, 2}, #2] &	1	False	False
0	p2	δ[{1, 2}, #2] &	1	True	False
0	p2	δ[{1, 1}, #2] + δ[{2, 2}, #2] &	1	True	False
0	p2	Total[#2] &	0	False	False
0	p2	randomUtility[2, 3, 3, 2]	0	False	False
0	p2	randomUtility[2, 3, 3, 280]	0	False	False
0	p2	randomUtility[2, 3, 3, 280865]	1	False	False
0.004	p2	randomUtility[2, 3, 3, 280865]	1	True	False
0.009	p2	randomUtility[2, 3, 3, 2]	0	True	False
0.014	p1	randomUtility[2, 3, 3, 280865]	1	False	False
0.019	p1	randomUtility[2, 3, 3, 280]	1	False	True
0.026	p1	δ[{1, 1}, #2] + δ[{1, 2}, #2] &	1	False	False
0.038	p2	δ[{1, 1}, #2] + δ[{2, 2}, #2] &	1	False	False
0.044	p1	randomUtility[2, 3, 3, 280]	1	True	True
0.051	p1	δ[{1, 2}, #2] + δ[{2, 3}, #2] &	1	False	False
0.052	p1	δ[{1, 1}, #2] + δ[{1, 2}, #2] &	1	True	False
0.059	p1	Times @@ #2 &	0	True	False
0.062	p2	Total[#2] &	0	True	False
0.066	p2	randomUtility[2, 3, 3, 280]	1	False	False
0.068	p1	randomUtility[2, 3, 3, 28086]	0	True	False
0.068	p1	randomUtility[2, 3, 3, 2]	0	False	True
0.071	p1	randomUtility[2, 3, 3, 28]	1	False	False
0.072	p2	randomUtility[2, 3, 3, 280]	0	True	False
0.077	p1	randomUtility[2, 3, 3, 280]	0	False	True
0.085	p1	randomUtility[2, 3, 3, 2808]	1	False	False
0.087	p2	randomUtility[2, 3, 3, 280]	1	True	False
0.092	p1	randomUtility[2, 3, 3, 28086]	1	False	False
0.094	p2	randomUtility[2, 3, 3, 280865]	0	False	False
0.109	p1	randomUtility[2, 3, 3, 28086]	1	True	False
0.139	p2	randomUtility[2, 3, 3, 2]	1	False	False
0.162	p1	randomUtility[2, 3, 3, 28]	1	True	False
0.165	p1	δ[{1, 2}, #2] + δ[{2, 3}, #2] &	1	True	False
0.165	p2	randomUtility[2, 3, 3, 2]	1	True	False
0.167	p1	transfer[2, 3][#2] &	1	False	False
0.171	p1	randomUtility[2, 3, 3, 280]	0	True	True
0.188	p2	δ[{1, 1}, #2] + δ[{2, 2}, #2] &	0	False	False
0.188	p2	δ[{1, 1}, #2] + δ[{2, 2}, #2] &	0	True	False
0.208	p1	randomUtility[2, 3, 3, 2808]	1	True	False
0.236	p1	Times @@ #2 &	1	False	False
0.244	p1	Total[#2] &	0	True	True
0.245	p1	Max[#2] &	0	False	True
0.245	p1	Max[#2] &	0	True	True
0.246	p2	Total[#2] &	1	False	False
0.246	p2	Total[#2] &	1	True	False
0.264	p2	randomUtility[2, 3, 3, 2808]	0	False	True

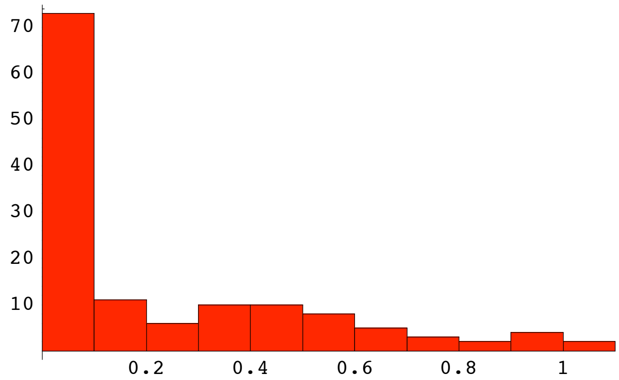
0.264	p2	randomUtility[2, 3, 3, 2808]	0	True	True
0.264	p2	randomUtility[2, 3, 3, 2808]	1	False	True
0.286	p1	randomUtility[2, 3, 3, 2]	0	True	True
0.298	p1	transfer[2, 3][#2] &	1	True	False
0.314	p2	randomUtility[2, 3, 3, 2808]	1	True	True
0.320	p1	Times @@ #2 &	1	True	False
0.344	p1	Total[#2] &	0	False	True
0.385	p2	δ[{1, 1}, #2] + δ[{1, 2}, #2] &	0	False	True
0.385	p2	δ[{1, 1}, #2] + δ[{1, 2}, #2] &	0	True	True
0.385	p2	δ[{1, 1}, #2] + δ[{1, 2}, #2] &	1	False	True
0.385	p2	δ[{1, 1}, #2] + δ[{1, 2}, #2] &	1	True	True
0.400	p1	randomUtility[2, 3, 3, 2]	1	False	True
0.490	p1	randomUtility[2, 3, 3, 2]	1	True	True
0.508	p2	transfer[2, 3][#2] &	0	False	True
0.508	p2	transfer[2, 3][#2] &	0	True	True
0.508	p2	transfer[2, 3][#2] &	1	False	True
0.508	p2	transfer[2, 3][#2] &	1	True	True
0.645	p1	Max[#2] &	1	False	True
0.685	p1	Max[#2] &	1	True	True
0.692	p2	δ[{1, 2}, #2] + δ[{2, 3}, #2] &	0	False	True
0.692	p2	δ[{1, 2}, #2] + δ[{2, 3}, #2] &	0	True	True
0.692	p2	Max[#2] &	0	False	True
0.692	p2	Max[#2] &	0	True	True
0.744	p1	Total[#2] &	1	False	True
0.800	p1	Total[#2] &	1	True	True
0.819	p2	randomUtility[2, 3, 3, 28]	0	False	True
0.819	p2	randomUtility[2, 3, 3, 28]	0	True	True
0.878	p2	randomUtility[2, 3, 3, 28]	1	False	True
0.878	p2	randomUtility[2, 3, 3, 28]	1	True	True
0.923	p2	randomUtility[2, 3, 3, 28086]	0	False	True
0.923	p2	randomUtility[2, 3, 3, 28086]	0	True	True
0.923	p2	randomUtility[2, 3, 3, 28086]	1	False	True
0.925	p2	randomUtility[2, 3, 3, 28086]	1	True	True
1	p2	δ[{1, 2}, #2] + δ[{2, 3}, #2] &	1	False	True
1	p2	δ[{1, 2}, #2] + δ[{2, 3}, #2] &	1	True	True
1	p2	Max[#2] &	1	False	True
1	p2	Max[#2] &	1	True	True



			horizon	dynamism	fuzzyQ
0	p1	Times @@ (#1 - 1 &) /@ #2 &	0	False	False
0	p1	Times @@ #2 &	0	False	False
0	p1	δ[{1, 2, 3}, #2] &	0	False	False
0	p1	δ[{1, 2, 3}, #2] &	0	True	False
0	p1	δ[{1, 2, 3}, #2] &	1	False	False
0	p1	δ[{1, 2, 3}, #2] &	1	True	False
0	p1	δ[{1, 2, 3}, #2] + δ[{3, 4, 5}, #2] + δ[{1, 5, 6}, #2] &	0	False	False
0	p1	δ[{1, 2, 3}, #2] + δ[{3, 4, 5}, #2] + δ[{1, 5, 6}, #2] &	0	True	False
0	p1	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] &	0	False	False
0	p1	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] &	0	True	False
0	p1	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] &	1	False	False
0	p1	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] &	1	True	False
0	p1	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] + δ[{1, 1, 3}, #2] + δ[{2, 2, 3}, #2] &	0	False	False
0	p1	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] + δ[{1, 1, 3}, #2] + δ[{2, 2, 3}, #2] &	0	True	False
0	p1	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] + δ[{1, 1, 3}, #2] + δ[{2, 2, 3}, #2] &	1	False	False

0	p1	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] + δ[{1, 1, 3}, #2] + δ[{2, 2, 3}, #2] &	1	True	False
0	p1	δ[{1, 2, 3}, #2] + δ[{2, 3, 4}, #2] &	0	False	False
0	p1	δ[{1, 2, 3}, #2] + δ[{2, 3, 4}, #2] &	0	True	False
0	p1	δ[{1, 2, 3}, #2] + δ[{2, 3, 4}, #2] &	1	False	False
0	p1	δ[{1, 2, 3}, #2] + δ[{2, 3, 4}, #2] &	1	True	False
0	p1	δ[{1, 2, 3}, #2] + δ[{4, 5, 6}, #2] &	0	False	False
0	p1	δ[{1, 2, 3}, #2] + δ[{4, 5, 6}, #2] &	0	True	False
0	p1	δ[{1, 2, 3}, #2] + δ[{4, 5, 6}, #2] &	1	False	False
0	p1	δ[{1, 2, 3}, #2] + δ[{4, 5, 6}, #2] &	1	True	False
0	p1	transfer[3, 6][#2] &	0	False	False
0	p1	randomUtility[3, 6, 3, 28]	0	False	False
0	p2	Times @@ (#1 - 1 &) /@ #2 &	0	False	False
0	p2	Times @@ (#1 - 1 &) /@ #2 &	0	True	False
0	p2	Times @@ #2 &	0	False	False
0	p2	δ[{1, 2, 3}, #2] &	0	False	False
0	p2	δ[{1, 2, 3}, #2] &	0	True	False
0	p2	δ[{1, 2, 3}, #2] &	1	False	False
0	p2	δ[{1, 2, 3}, #2] &	1	True	False
0	p2	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] &	0	False	False
0	p2	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] &	1	True	False
0	p2	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] + δ[{1, 1, 3}, #2] + δ[{2, 2, 3}, #2] &	0	False	False
0	p2	δ[{1, 2, 3}, #2] + δ[{2, 3, 4}, #2] &	0	False	False
0	p2	δ[{1, 2, 3}, #2] + δ[{2, 3, 4}, #2] &	0	True	False
0	p2	δ[{1, 2, 3}, #2] + δ[{4, 5, 6}, #2] &	0	False	False
0	p2	δ[{1, 2, 3}, #2] + δ[{4, 5, 6}, #2] &	0	True	False
0	p2	δ[{1, 2, 3}, #2] + δ[{4, 5, 6}, #2] &	1	False	False
0	p2	δ[{1, 2, 3}, #2] + δ[{4, 5, 6}, #2] &	1	True	False
0	p2	randomUtility[3, 6, 3, 28086]	0	False	False
0.003	p2	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] &	1	False	False
0.003	p1	Times @@ #2 &	0	True	False
0.004	p1	δ[{1, 2, 3}, #2] + δ[{3, 4, 5}, #2] + δ[{1, 5, 6}, #2] &	1	False	False
0.004	p2	Times @@ #2 &	0	True	False
0.005	p1	Times @@ (#1 - 1 &) /@ #2 &	0	True	False
0.005	p2	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] &	0	True	False
0.007	p2	randomUtility[3, 6, 3, 28086]	0	True	False
0.007	p1	transfer[3, 6][#2] &	0	True	False
0.013	p1	randomUtility[3, 6, 3, 2]	0	False	True
0.014	p1	δ[{1, 2, 3}, #2] + δ[{3, 4, 5}, #2] + δ[{1, 5, 6}, #2] &	1	True	False
0.015	p1	randomUtility[3, 6, 3, 2]	0	True	True
0.028	p2	δ[{1, 2, 3}, #2] + δ[{3, 4, 5}, #2] + δ[{1, 5, 6}, #2] &	1	False	True
0.037	p1	randomUtility[3, 6, 3, 2808]	0	False	True
0.049	p1	randomUtility[3, 6, 3, 280]	0	False	True
0.049	p2	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] + δ[{1, 1, 3}, #2] + δ[{2, 2, 3}, #2] &	1	True	False
0.053	p2	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] + δ[{1, 1, 3}, #2] + δ[{2, 2, 3}, #2] &	0	True	False
0.057	p1	randomUtility[3, 6, 3, 280]	0	True	True
0.062	p2	δ[{1, 1, 1}, #2] + δ[{2, 2, 2}, #2] + δ[{1, 1, 3}, #2] + δ[{2, 2, 3}, #2] &	1	False	False
0.064	p1	randomUtility[3, 6, 3, 2808]	0	True	True
0.065	p1	Total[#2] &	0	True	True
0.065	p2	δ[{1, 2, 3}, #2] + δ[{3, 4, 5}, #2] + δ[{1, 5, 6}, #2] &	0	False	True
0.065	p2	δ[{1, 2, 3}, #2] + δ[{3, 4, 5}, #2] + δ[{1, 5, 6}, #2] &	0	True	True
0.068	p1	randomUtility[3, 6, 3, 28]	0	True	False
0.072	p2	δ[{1, 2, 3}, #2] + δ[{3, 4, 5}, #2] + δ[{1, 5, 6}, #2] &	1	True	True
0.074	p2	Times @@ #2 &	1	False	False
0.076	p2	Times @@ (#1 - 1 &) /@ #2 &	1	False	False
0.080	p1	transfer[3, 6][#2] &	1	False	False
0.084	p2	randomUtility[3, 6, 3, 280]	0	False	True
0.089	p1	Total[#2] &	0	False	True
0.095	p2	Total[#2] &	0	True	True
0.102	p1	randomUtility[3, 6, 3, 28086]	0	False	True
0.105	p2	transfer[3, 6][#2] &	0	False	True
0.108	p2	Total[#2] &	0	False	True
0.114	p1	randomUtility[3, 6, 3, 28086]	0	True	True

0.118	p2	Times @@ (#1 - 1 &) /@ #2 &	1	True	False
0.126	p2	δ[{1, 2, 3}, #2] + δ[{2, 3, 4}, #2] &	1	False	False
0.142	p2	Times @@ #2 &	1	True	False
0.151	p2	transfer[3, 6][#2] &	0	True	True
0.165	p1	transfer[3, 6][#2] &	1	True	False
0.199	p2	δ[{1, 2, 3}, #2] + δ[{2, 3, 4}, #2] &	1	True	False
0.207	p2	randomUtility[3, 6, 3, 280]	0	True	True
0.218	p2	transfer[3, 6][#2] &	1	False	True
0.257	p1	randomUtility[3, 6, 3, 28086]	1	False	True
0.263	p2	transfer[3, 6][#2] &	1	True	True
0.285	p1	Times @@ (#1 - 1 &) /@ #2 &	1	True	False
0.286	p1	randomUtility[3, 6, 3, 280865]	1	False	True
0.302	p1	randomUtility[3, 6, 3, 2808]	1	False	True
0.316	p1	Max[#2] &	0	False	?
0.325	p1	randomUtility[3, 6, 3, 28]	1	False	False
0.326	p1	randomUtility[3, 6, 3, 2]	1	False	True
0.345	p1	randomUtility[3, 6, 3, 280]	1	False	True
0.347	p1	randomUtility[3, 6, 3, 280865]	0	True	True
0.351	p1	randomUtility[3, 6, 3, 280865]	1	True	True
0.365	p1	Times @@ #2 &	1	False	False
0.372	p1	randomUtility[3, 6, 3, 280865]	0	False	True
0.390	p2	Max[#2] &	0	False	?
0.413	p1	randomUtility[3, 6, 3, 28086]	1	True	True
0.422	p1	randomUtility[3, 6, 3, 2]	1	True	True
0.430	p2	randomUtility[3, 6, 3, 28086]	1	False	False
0.433	p2	randomUtility[3, 6, 3, 2808]	0	True	True
0.433	p1	randomUtility[3, 6, 3, 2808]	1	True	True
0.467	p2	Total[#2] &	1	False	True
0.479	p1	randomUtility[3, 6, 3, 28]	1	True	False
0.484	p2	randomUtility[3, 6, 3, 28086]	1	True	False
0.496	p2	randomUtility[3, 6, 3, 280865]	1	False	True
0.497	p2	randomUtility[3, 6, 3, 2808]	0	False	True
0.513	p2	Total[#2] &	1	True	True
0.539	p1	randomUtility[3, 6, 3, 280]	1	True	True
0.540	p2	randomUtility[3, 6, 3, 2]	0	False	True
0.549	p2	randomUtility[3, 6, 3, 280865]	1	True	True
0.560	p1	Times @@ #2 &	1	True	False
0.577	p2	randomUtility[3, 6, 3, 2]	0	True	True
0.586	p2	randomUtility[3, 6, 3, 2808]	1	False	True
0.586	p2	randomUtility[3, 6, 3, 28]	1	False	True
0.606	p2	randomUtility[3, 6, 3, 280]	1	False	True
0.614	p2	randomUtility[3, 6, 3, 28]	1	True	True
0.641	p2	randomUtility[3, 6, 3, 2808]	1	True	True
0.655	p2	randomUtility[3, 6, 3, 280]	1	True	True
0.690	p2	randomUtility[3, 6, 3, 2]	1	False	True
0.742	p2	randomUtility[3, 6, 3, 28]	0	False	True
0.763	p2	randomUtility[3, 6, 3, 2]	1	True	True
0.797	p2	randomUtility[3, 6, 3, 28]	0	True	True
0.824	p2	randomUtility[3, 6, 3, 280865]	0	False	True
0.850	p2	randomUtility[3, 6, 3, 280865]	0	True	True
0.911	p1	Total[#2] &	1	False	True
0.950	p1	Max[#2] &	1	False	?
0.950	p1	Max[#2] &	1	True	?
0.992	p1	Total[#2] &	1	True	True
1.032	p2	Max[#2] &	1	False	?
1.032	p2	Max[#2] &	1	True	?
?	p1	Max[#2] &	0	True	?
?	p2	Max[#2] &	0	True	?



8.2.3. Cases of harmul serendipity

```
<< strategyUtility.txt; << randomStrategyUtility.txt; << maxStrategyUtility.txt;
```

Check:

```
With[{player = "p1", d = 2, f = 3, j = 3}, With[{fateTree1 = fateTree[player, d, f, j],
   utility = randomUtility[d, f, j, 280865], horizon = 1, dynamism = True},
  Block[{strategyUtility, randomStrategyUtility, maxStrategyUtility, no, yes, zero, unit},
   make[fateTree1, probability]@strategyUtility[_, _, _, _, utility, False, horizon, dynamism];
   Print[1. (no = strategyUtility[player, d, f, j, utility, False, horizon, dynamism])]; make[
     fateTree1, probability]@strategyUtility[_, _, _, _, utility, dynamism, horizon, dynamism];
   Print[1. (yes = strategyUtility[player, d, f, j, utility, True, horizon, dynamism])];
   make@randomStrategyUtility[fateTree1, utility];
   Print[1. (zero = randomStrategyUtility[player, d, f, j, utility])];
   make@maxStrategyUtility[fateTree1, utility];
   Print[1. (unit = maxStrategyUtility[player, d, f, j, utility])];
   1. Last@# - First@# &@{(no - zero)/(unit - zero), (yes - zero)/(unit - zero)}]]]
```

```
0.635986

0.635171

0.426232

0.643238

-0.00375516
```

```
With[{player = "p2", d = 2, f = 3, j = 3}, With[{fateTree1 = fateTree[player, d, f, j],
   utility = randomUtility[d, f, j, 280865], horizon = 0, dynamism = True},
  Block[{strategyUtility, randomStrategyUtility, maxStrategyUtility, no, yes, zero, unit},
   make[fateTree1, probability]@strategyUtility[_, _, _, _, utility, False, horizon, dynamism];
   Print[1. (no = strategyUtility[player, d, f, j, utility, False, horizon, dynamism])]; make[
     fateTree1, probability]@strategyUtility[_, _, _, _, utility, dynamism, horizon, dynamism];
   Print[1. (yes = strategyUtility[player, d, f, j, utility, True, horizon, dynamism])];
   make@randomStrategyUtility[fateTree1, utility];
   Print[1. (zero = randomStrategyUtility[player, d, f, j, utility])];
   make@maxStrategyUtility[fateTree1, utility];
   Print[1. (unit = maxStrategyUtility[player, d, f, j, utility])];
   1. Last@# - First@# &@{(no - zero)/(unit - zero), (yes - zero)/(unit - zero)}]]]
```

```
0.430143

0.427943

0.339934

0.431121

-0.0241287
```

What is going on can be seen easily, as parameters are small and strategy is Markovian.

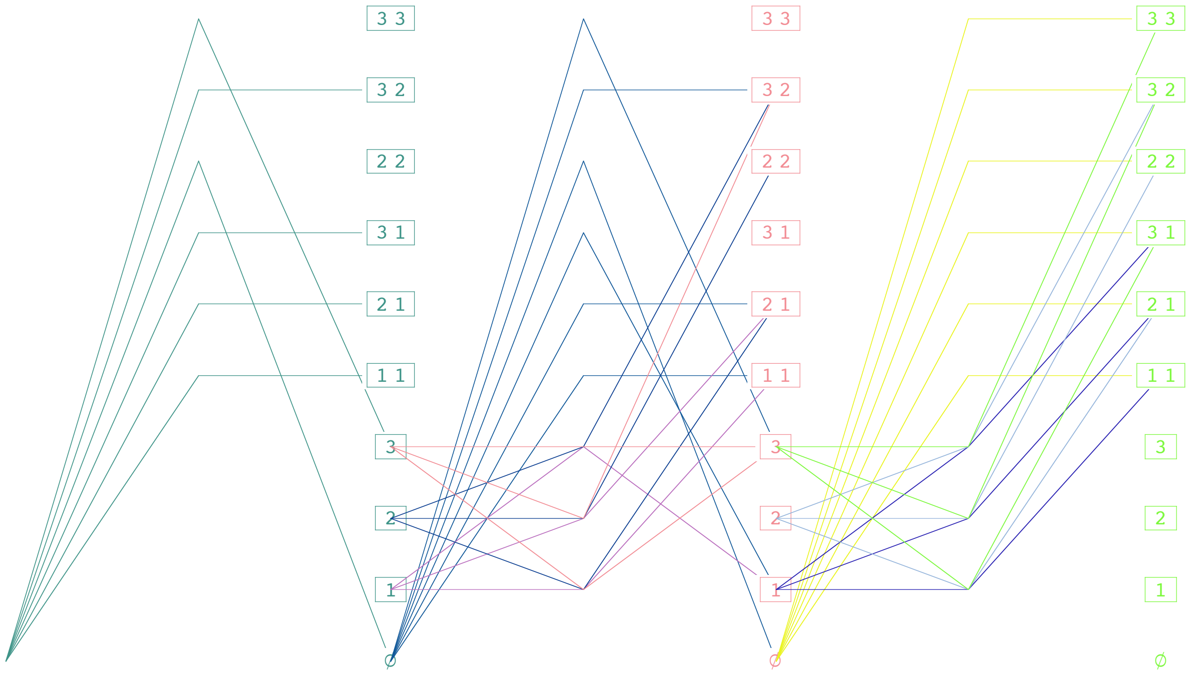
```
player = "p1"; {d = 2, f = 3, j = 3}; utility = randomUtility[d, f, j, 280865];
fateTree1 = fateTree["p1", d, f, j, utility];
strategy1 = strategy[fateTree1, utility, probability, True, 1, True]
```

```
{{0, {{{{}, {{{1, 1}, {{{1, 1}, {5}}}}, {{1, 2}, {{{1, 2}, {6}}}}, {{1, 3}, {{{1, 3}, {7}}}},
      {{2, 2}, {{{}, {1}}}}, {{2, 3}, {{{2, 3}, {9}}}}, {{3, 3}, {{{3}, {4}}}}}}}, {}}},
 {1, {{{{}, {{{1, 1}, {{{1, 1}, {5}}}}, {{1, 2}, {{{1, 2}, {6}}}}, {{1, 3}, {{{1}, {2}}}},
      {{2, 2}, {{{}, {1}}}}, {{2, 3}, {{{2, 3}, {8}}}}, {{3, 3}, {{{3}, {4}}}}}},
    {{1}, {{{1}, {{{1, 1}, {5}}}}, {{2}, {{{1, 2}, {6}}}}, {{3}, {{{1}, {2}}}}}},
    {{2}, {{{1}, {{{1, 2}, {6}}}}, {{2}, {{{2, 2}, {7}}}}, {{3}, {{{2, 3}, {8}}}}}},
    {{3}, {{{1}, {{{3}, {4}}}}, {{2}, {{{2, 3}, {8}}}}, {{3}, {{{3}, {4}}}}}}},
   {{{1, 1}, {}}, {{1, 2}, {}}, {{1, 3}, {}}, {{2, 2}, {}}, {{2, 3}, {}}}}},
 {2, {{{{}, {{{1, 1}, {{{1, 1}, {1}}}}, {{1, 2}, {{{1, 2}, {2}}}}, {{1, 3}, {{{1, 3}, {3}}}},
      {{2, 2}, {{{2, 2}, {4}}}}, {{2, 3}, {{{2, 3}, {5}}}}, {{3, 3}, {{{3, 3}, {6}}}}}},
    {{1}, {{{1}, {{{1, 1}, {1}}}}, {{2}, {{{1, 2}, {2}}}}, {{3}, {{{1, 3}, {3}}}}}},
    {{2}, {{{1}, {{{1, 2}, {2}}}}, {{2}, {{{2, 2}, {4}}}}, {{3}, {{{2, 3}, {5}}}}}},
    {{3}, {{{1}, {{{1, 3}, {3}}}}, {{2}, {{{2, 3}, {5}}}}, {{3}, {{{3, 3}, {6}}}}}}},
   {{{1, 1}, {}}, {{1, 2}, {}}, {{2, 2}, {}}, {{2, 3}, {}}, {{3, 3}, {}}}}},
 {3, {{}, {{{1, 1}, {}}, {{1, 2}, {}}, {{1, 3}, {}}, {{2, 2}, {}}, {{2, 3}, {}}, {{3, 3}, {}}}}}}
```

```
1. initial@utilityAndStrategy[fateTree1, utility]@strategy1
```

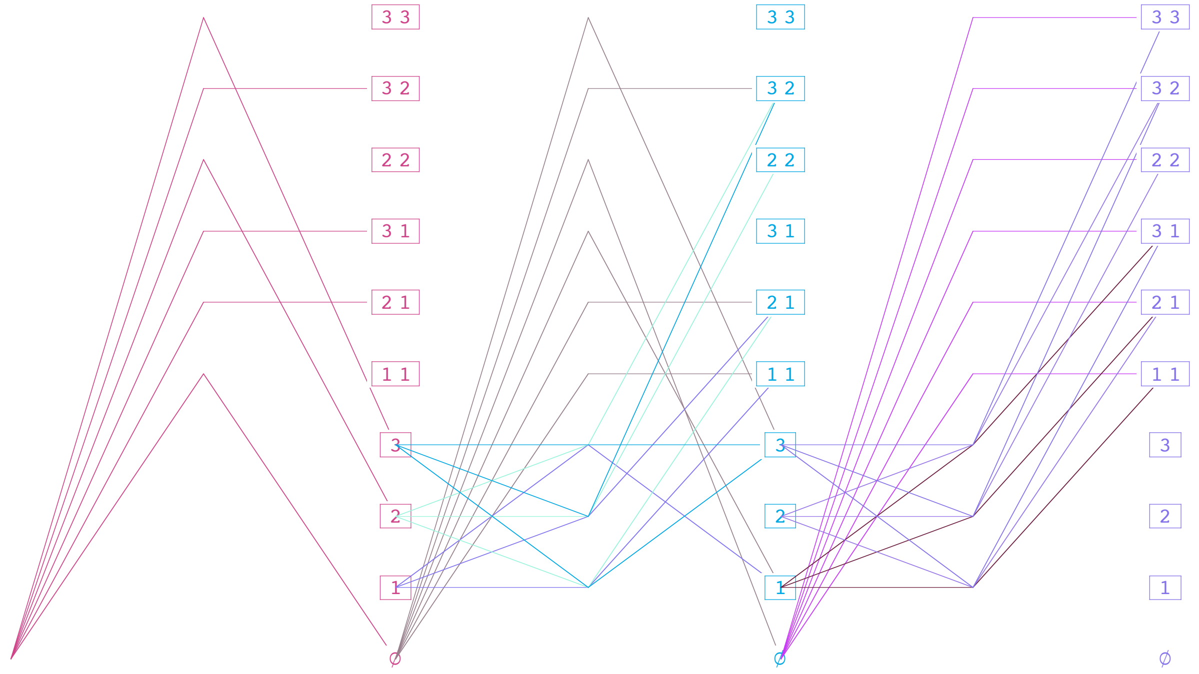
```
0.635171
```

```
Show[strategyGraphics[fateTree1, withProbabilities → False]@strategy1, ImageSize → 8 72];
```



Compare with complete most useful strategy.

```
strategy2 = extractStrategy@maxUtilityAndStrategy[fateTree1, utility];
Show[strategyGraphics[fateTree1, withProbabilities → False]@strategy2, ImageSize → 8 72];
```



Strategies occur to differ by the choice of 11 after first cast! ⚠ Missing: diff tool (see AuthorTools).

```
strategy1 == strategy2
Rest@strategy1 == Rest@strategy2
First /@ {strategy1, strategy2}
```

```
False

True

{{0, {{{{}, {{{1, 1}, {{{1, 1}, {5}}}}, {{1, 2}, {{{1, 2}, {6}}}}, {{1, 3}, {{{1, 3}, {7}}}},
      {{2, 2}, {{{}, {1}}}}, {{2, 3}, {{{2, 3}, {9}}}}, {{3, 3}, {{{3}, {4}}}}}}}}, {}}},
 {0, {{{{}, {{{1, 1}, {{{}, {1}}}}, {{1, 2}, {{{1, 2}, {6}}}}, {{1, 3}, {{{1, 3}, {7}}}},
      {{2, 2}, {{{2}, {3}}}}, {{2, 3}, {{{2, 3}, {9}}}}, {{3, 3}, {{{3}, {4}}}}}}}}, {}}}}
```

```
allCombinations[d, f]
1. utility[1, #] & /@ allCombinations[d, f]
```

```
{{1, 1}, {1, 2}, {1, 3}, {2, 2}, {2, 3}, {3, 3}}

{0.50495, 0.643564, 0.712871, 0.19802, 0.693069, 0.019802}
```

```
player =.; utility =.; fateTree1 =.; strategy1 =.; strategy2 =.;

Clear@strategyUtility; Clear@randomStrategyUtility; Clear@maxStrategyUtility;
```

8.2.4. Serendipity makes horizon and dynamism useful

8.2.5. No serendipity makes horizon or dynamism often strictly harmful, especially for fuzzy utility functions

8.2.6. For non-fuzzy utility functions, horizon and dynamism are often more useful than serendipity

8.3. Meta-policy

Def.: a *meta-policy* is a program evaluating exactly one policy.

A typical meta-policy consists in choosing among a finite set of policies, according to some utility function.

Def.: the *meta-timing* (resp. *spacing*) of a policy is the timing (resp. spacing) of its judging for some utility function.

The timing of the typical meta-policy is the sum of meta-timings on the policy set:
"timing of meta-policy = sum of policy meta-timings".
(We neglect the timing of utility maximizing.)

As `mean-max` judges itself, remarkably, its meta-timing (resp. spacing) identifies with its timing (resp. spacing). Indeed, among all "reasonable" policies `mean-max` maximizes timing but minimizes meta-timing [1]; we will checked this numerically:

```
player = "p1"; {d = 3, f = 6, j = 3}; utility = only@Cases[utilityList[d, f, j], transfer[__]@_ &];
fateTree1 = fateTree[player, d, f, j]; fateTree2 = fateTree[player, d, f, j, utility];
```

```
Block[{maxStrategyUtility}, Timing@make@maxStrategyUtility[fateTree1, utility]]
timing1 = First@%;
```

$\{0.34\ \text{Second},\ \frac{349621}{93312}\}$

```
Block[{maxStrategyUtility}, Timing@make@maxStrategyUtility[fateTree2, utility]]
```

$\{0.26\ \text{Second},\ \frac{349621}{93312}\}$

Some meta-timings are lower than `mean-max` timing, because of a goal utility function short cut (using compiled probabilities).

```
scanPrint[
 timing2 = Block[{strategyUtility}, Distribute[{{False, True}, {0, 1}, {False, True}}, List, List,
    List, {##, Timing@make[fateTree2, probability]@strategyUtility[_, _, _, _, utility, ##]} &]]]
```

$\{\text{False}, 0, \text{False}, \{0.03\ \text{Second}, \frac{952727}{279936}\}\}$

$\{\text{False}, 0, \text{True}, \{6.94\ \text{Second}, \frac{952727}{279936}\}\}$

$\{\text{False}, 1, \text{False}, \{0.26\ \text{Second}, \frac{478337}{139968}\}\}$

$\{\text{False}, 1, \text{True}, \{0.41\ \text{Second}, \frac{52757}{15552}\}\}$

$\{\text{True}, 0, \text{False}, \{0.46\ \text{Second}, \frac{952727}{279936}\}\}$

$\{\text{True}, 0, \text{True}, \{6.24\ \text{Second}, \frac{956723}{279936}\}\}$

$\{\text{True}, 1, \text{False}, \{0.75\ \text{Second}, \frac{167123}{46656}\}\}$

$\{\text{True}, 1, \text{True}, \{1.45\ \text{Second}, \frac{58057}{15552}\}\}$

```
Apply[And, #[[-1, 1]] ≥ timing1 /. Second → 1 & /@ timing2]
```

```
False
```

Without short cut:

```
scanPrint[
 timing3 = Block[{strategyUtility}, Distribute[{{False, True}, {0, 1}, {False, True}}, List,
    List, List, {##, Timing@make[fateTree2, probability, meanMaxShortCut → False]@
        strategyUtility[_, _, _, _, utility, ##]} &]]]
```

$\{\text{False, 0, False, }\{0.88\ \text{Second}, \frac{952727}{279936}\}\}$

$\{\text{False, 0, True, }\{7.06\ \text{Second}, \frac{952727}{279936}\}\}$

$\{\text{False, 1, False, }\{1.44\ \text{Second}, \frac{478337}{139968}\}\}$

$\{\text{False, 1, True, }\{0.42\ \text{Second}, \frac{52757}{15552}\}\}$

$\{\text{True, 0, False, }\{1.3\ \text{Second}, \frac{952727}{279936}\}\}$

$\{\text{True, 0, True, }\{6.23\ \text{Second}, \frac{956723}{279936}\}\}$

$\{\text{True, 1, False, }\{1.75\ \text{Second}, \frac{167123}{46656}\}\}$

$\{\text{True, 1, True, }\{1.48\ \text{Second}, \frac{58057}{15552}\}\}$

Q. E. D. ⚠ Try more utility functions.

```
Apply[And, #[[-1, 1]] ≥ timing1 /. Second → 1 & /@ timing3]
```

```
True
```

```
player =.; {d, f, j} =.; utility =.; fateTree1 =.;
fateTree2 =.; timing1 =.; timing2 =.; timing3 =.;
```

In general, policy utility should take into account not only strategy utility but also *policy timing and spacing* (negatively). However, timing and spacing are not yet available, as goal-driven policies were judged but not realized.

9. References

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