Analytical solutions of elastic regions weakened by elliptic cracks

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The present study deals with the analytic resolution of fundamental problems of the plane elasticity theory. Based on the general framework suggested by Kolossov and Muskhelishvili through the complex representation theory, the investigation focuses on the determination of the analytical expressions of stress and displacements fields inside an elastic plate containing elliptic cracks and subjected to specific boundary conditions. This analytic approach, initially difficult to perform, is associated to Mathematica to overcome computations difficulties arising when one attempts calculate exact analytical solutions.
The obtained results precise the validity of asymptotic solutions, widely used in fracture mechanics. Some specific aspects of the fracture mechanics such as stress contourlines at the crack's vicinity and the stress intensity factor are analyzed. Moreover, the more complex case of two elliptic cracks is also treated and corresponding analytic solutions provided.

■ 1. Introduction

To solve fundamental problems of the plane elasticity theory, some numerical techniques such as the finite elements methods, are often used. A few investigations has been devoted to analytic approaches of these particular problems but all them generally propose approximative semi-analytic solutions.
For the complete solution of equations of plane elasticity theory, the complex representation introduced by G.V. Kolossov [5] and developed by N. I. Muskhelishvili [6] has been applied by several authors to study problems of plate containing holes and to suggest numerical and semi-numerical results: Sneddon [8], Green and Zerna [4], Aifantis et al. [1]. These initial investigations give up, in front of the complexity of analytical calculations, to clarify the solutions and being generally satisfied to give partial or numerical results.

2. The plane elasticity theory

In the plane theory of elasticity and in the assumption of vanishing body forces, stress components can be expressed by means of the so-called Airy function $\mathcal{A}(x, y)$ defined by:

$$\sigma_{xx} = \mathcal{A}_{,yy}\ ,\quad \sigma_{yy} = \mathcal{A}_{,xx}\ ,\quad \tau_{xy} = -\mathcal{A}_{,xy} \tag{1}$$

The Airy function, which is biharmonic, have also to satisfy to boundary and compatibility conditions of the specific problem. According to the theorem of E. Almansi, any poly-harmonic function of N order is representable by N harmonic functions. The solution of plane elasticity problems therefore, rises from two functions (φ and χ) of the complex variable $z = x + i\,y$.

$$\mathcal{A}(\mathrm{z}) = \mathrm{Re}\,(\bar{z}\,\varphi(z) + \chi(z)) \tag{2}$$

where Re denotes the real part and $\bar{z} = x - i\,y$ the conjugate of the compex variable z. Boundary conditions on the hole contour can be expressed as:

$$f\,(x,\ y) = i\,(X + i\,Y) + \mathrm{const} = \varphi\,(z) + \overline{z\varphi'\,(z)} + \overline{\psi\,(z)} \tag{3}$$

(X, Y) are the components of the resultant force acting on hole's contours in the case of multiply connected regions and $\psi(z) = \chi'(z)$. Kolossov-Muskhelishvili potentials (φ and χ) depend on the geometry of the selected region: finite or infinite, simply or multiply connected. Subsequent stress and displacements fields are deduced from the following equations:

$$\begin{aligned}
&\sigma_{xx} + \sigma_{yy} = 2\,\mathrm{Re}(\phi(z)) \\
&\sigma_{yy} - \sigma_{xx} + 2\,i\,\tau_{xy} = 2\,(\bar{z}\,\phi'(z) + \psi'(z)) \\
&2\,\mu\,(u_x + i\,u_y) = \kappa\,\varphi(z) - z\,\overline{\phi(z)} - \psi(z) \\
&\kappa = \tfrac{5\,\lambda+6\,\mu}{3\,\lambda+2\,\mu} \qquad \text{in a plane stress state} \\
&\kappa = \tfrac{\lambda+3\,\mu}{\lambda+\mu} \qquad \text{in a plane strain state}
\end{aligned} \tag{4}$$

Where $\phi(z) = \varphi'(z)$ and (μ, λ), the Lamé constants of the isotropic elastic body.

Let's consider an elastic region D containing n elliptic cracks $L_k = a_k\,b_k$, where a_k and b_k are the crack tips of the selected crack L_k lying on the x-coordinate.

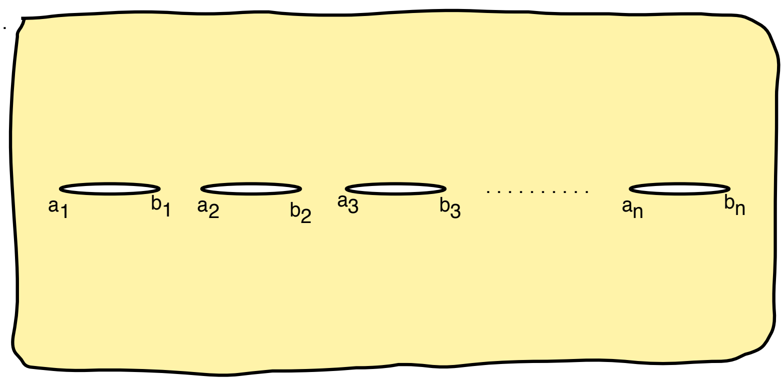


Figure 1.

This elastic region is subjected to a finite remote stress at infinity. Through the general framework suggested by Muskhilshvili, the complex stress functions associated to this problem are holomorphic on the region exterior to the considered cracks and the following formulas hold true:

$$\begin{aligned}
&\phi(z) = -i\,\frac{X+iY}{2\pi(\kappa+1)}\,\frac{1}{z} + \Gamma_1 + 0\left(\frac{1}{z^2}\right)\\
&\Psi(z) = \kappa\,\frac{X-iY}{2\pi(\kappa+1)}\,\frac{1}{z} + \Gamma_2 + 0\left(\frac{1}{z^2}\right)\\
&\Gamma_1 = \frac{N_1+N_2}{4} + i\,\frac{2\mu\varepsilon_\infty}{\kappa+1}\\
&\Gamma_2 = \frac{N_2-N_1}{2}\,e^{-2i\alpha}
\end{aligned} \tag{5}$$

$(N_1,\ N_2)$ are the principal values of the remote stress applied at infinity, $\alpha = (N_1,\, O\,x)$ and ε_∞ the rotation at infinity which can be assumed to vanish without influencing the generality of results. Let's replace the function $\Psi(z)$ by a new one, $\Omega(z)$ defined by:

$$\Omega(z) = \bar{\phi}(z) + z\,\overline{\phi'}(z) + \bar{\Psi}(z) = \overline{\Gamma_1} + \overline{\Gamma_2} + \kappa\,\frac{X+iY}{2\pi(\kappa+1)}\,\frac{1}{z} + 0\left(\frac{1}{z^2}\right) \tag{6}$$

For the problem of infinite elastic region containing n elliptic cracks, following alternative expressions are suggested:

$$\begin{aligned}
&\phi(z) = \frac{P_n(z)}{\chi(z)} - \frac{1}{2}\,\overline{\Gamma_2}\\
&\Omega(z) = \frac{P_n(z)}{\chi(z)} + \frac{1}{2}\,\overline{\Gamma_2}\\
&\chi(z) = \prod_{k=1}^{n}\sqrt{(z-a_k)(z-b_k)}\,,\\
&P_n(z) = c_0\,z^n + c_1\,z^{n-1} + \ldots + c_n
\end{aligned} \tag{7}$$

Boundary conditions at infinity lead to the following relations:

$$\varphi(\infty) = \Gamma \quad \Longrightarrow c_0 = \Gamma_1 + \frac{1}{2}\,\overline{\Gamma_2} \tag{8}$$

For simplicity, the elliptic cracks contours are assumed to be stress-free:

$$X = Y = 0 \quad \Longrightarrow \int_{L_k} \frac{P_n(t)}{\chi(z)}\,dt;\ k = 1, 2, \ \ldots,\ n \tag{9}$$

The other parameters $c_1,\ c_2,\ \ldots,\ c_n$ of the function $P_n(z)$ are then calculated through the resolution of the equations (9). Stress parameters Γ_1 and Γ_2 are known since the remote stress at infinity is selected (equations 5c and 5d). Moreover, the process of resolution can be resumed as the following:

1. Calculation of Γ_1 and Γ_2 through (5) according to the stress boundary conditions at infinity;
2. Calculation of constants $c_1,\ c_2,\ \ldots,\ c_n$ from system (9);
3. Expression of stress functions defined using (7);
4. Stress and displacement fields' expression by (4).

$$\begin{aligned}
&\sigma_{xx} + \sigma_{yy} = 4\,\mathrm{Re}(\phi(z)),\\
&\sigma_{yy} - \sigma_{xx} = 2\,\mathrm{Re}((\bar{z} - z)\phi'(z) + \bar{\Omega}(z) - \phi(z))\\
&\sigma_{xy} = \mathrm{Im}((\bar{z} - z)\phi'(z) + \bar{\Omega}(z) - \phi(z))\\
&2\mu\,(u_x + i\,u_y) = \kappa\,\varphi(z) - z\,\overline{\varphi'(z)} - \overline{\psi(z)}
\end{aligned} \tag{10}$$

The proposed procedure is applied to the problem of elastic infinite plate weakened by one or two elliptic cracks and subjected to particular remote laodings.

□ 1. Infinite plate with one elliptic crack sujected to a Mode I loading.

For this classical case, boundary conditions at infinity ($\sigma_{yy} = \sigma^\infty,\ \sigma_{xx} = \tau_{xy} = 0$) lead to:

$$N_1 = 0\,,\ N_2 = \sigma,\ \ \alpha = 0 \quad \Longrightarrow \Gamma_1 = \frac{\sigma^\infty}{4}\,,\ \Gamma_2 = \frac{\sigma^\infty}{2} \tag{11}$$

On the other hand, crack tips are defined by $a_1 = 0$ and $b_1 = -2a$, so:

$$\chi(z) = \sqrt{z(z+2a)}, \text{ and } P_1(z) = c_0 z + c_1 \quad (12)$$

c_0 is given by (8) and c_1 calculated from the equation $\int_0^{-2a} \frac{C_0 t + C_1}{\sqrt{t(t+2a)}} dt = 0$ leading to $c_1 = a\, c_0$. Finally the stress functions expressions are:

$$\begin{aligned} \phi(z) &= \frac{\sigma^\infty}{4}\left(-1 + \frac{2(z+a)}{\sqrt{z(z+2a)}}\right); \\ \Omega(z) &= \frac{\sigma^\infty}{4}\left(1 + \frac{2(z+a)}{\sqrt{z(z+2a)}}\right); \end{aligned} \quad (13)$$

The analytical expressions of stress tensor components then rise for relations (4). The following function is defined to perform the calculation of the stress components:

```
CrackStress[N1_, N2_, α_] := Module[{v2, x, y, c0, sol, Γ1, Γ2},
  polarForm[expr_] := expr /. z → r Exp[I θ];
  conjugue[expr_] := expr /. Complex[x_, y_] → Complex[x, -y];
  realPart[expr_] := (expr + conjugue[expr]) / 2;
  imaginaryPart[expr_] := (expr - conjugue[expr]) / (2 I);
  sol = Solve[Integrate[(c1 + c0 z)/Sqrt[z (z + 2 a)], {z, -2 a, 0},
       Assumptions → a > 0] == 0, c1] // Flatten;
  Γ1 = (N1 + N2)/4; Γ2 = (N2 - N1)/2 Exp[-2 I α]; c0 = Γ1 + conjugue[Γ2]/2;
  ϕ = ((c0 z + c1)/Sqrt[z (z + 2 a)] - conjugue[Γ2]/2) /. sol;
  Ω = ((c0 z + c1)/Sqrt[z (z + 2 a)] + conjugue[Γ2]/2) /. sol;
  Φ = D[ϕ, z]; Ψ = Ω - ϕ - z Φ;
  v2 = r Exp[-I θ] polarForm[Φ] + polarForm[Ψ];
  stress = LinearSolve[{{1, 1}, {-1, 1}},
    {4 realPart[polarForm[ϕ]], 2 realPart[v2]}];
  σxx = σxx = ComplexExpand[stress[[1]],
     TargetFunctions -> {Re, Im}];
  σyy = ComplexExpand[stress[[2]], TargetFunctions -> {Re, Im}];
  σxy = ComplexExpand[imaginaryPart[v2],
    TargetFunctions -> {Re, Im}];
 ];
```

For a mode I loading, one can run CrackStress[0,σ_∞,0] and obtains after simplifications the following expressions:

$$\begin{aligned}
\sigma_{\mathrm{xx}} = {} & \frac{1}{2\sqrt{r}\,(4a^2+4r\cos(\theta)a+r^2)^{9/4}} \\
& \Big(\sigma_\infty\Big(-16a(4a^2+r^2)\cos(\theta)\sqrt[4]{4a^2+4r\cos(\theta)a+r^2}\;r^{3/2} - \\
& 32a^2\sqrt[4]{4a^2+4r\cos(\theta)a+r^2}\;r^{5/2} - 16a^2\sqrt[4]{4a^2+4r\cos(\theta)a+r^2} \\
& \cos(2\theta)\,r^{5/2} - 2\sqrt[4]{4a^2+4r\cos(\theta)a+r^2}\;r^{9/2} + 4\cos(\\
& \theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta)))\,r^5 + \\
& 20a\cos(\tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r^4 - 2\sqrt{4a^2+4r\cos(\theta)a+r^2}\;\cos(\theta - \\
& \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta)))\,r^4 + 16a \\
& \cos(2\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r^4 - 8a\sqrt{4a^2+4r\cos(\theta)a+r^2}\;\cos(\\
& \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta)))\,r^3 + 32a^2 \\
& \cos(\theta + \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r^3 + 80a^2 \\
& \cos(\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r^3 - 6a\sqrt{4a^2+4r\cos(\theta)a+r^2} \\
& \cos(2\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r^3 + 16a^2\cos(\\
& 3\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta)))\,r^3 + \\
& 128a^3\cos(\tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r^2 + 16a^3 \\
& \cos(\tfrac{1}{2}(4\theta + \tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta)))) \\
& r^2 - 8a^2\sqrt{4a^2+4r\cos(\theta)a+r^2} \\
& \cos(\theta + \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r^2 - 24a^2\sqrt{4a^2+4r\cos(\theta)a+r^2} \\
& \cos(\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r^2 + 80a^3 \\
& \cos(2\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r^2 - 5a^2\sqrt{4a^2+4r\cos(\theta)a+r^2} \\
& \cos(3\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r^2 + a^2\sqrt{4a^2+4r\cos(\theta)a+r^2} \\
& \cos(5\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r^2 - 24a^3\sqrt{4a^2+4r\cos(\theta)a+r^2} \\
& \cos(\tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta)))\,r + 64 \\
& a^4\cos(\theta + \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r + 128a^4 \\
& \cos(\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta)))\,r - \\
& 20a^3\sqrt{4a^2+4r\cos(\theta)a+r^2} \\
& \cos(2\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r + 4a^3\sqrt{4a^2+4r\cos(\theta)a+r^2} \\
& \cos(4\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) \\
& r - 32a^4\sqrt[4]{4a^2+4r\cos(\theta)a+r^2}\;\sqrt{r} + \\
& 64a^5\cos(\tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) - \\
& 20a^4\sqrt{4a^2+4r\cos(\theta)a+r^2} \\
& \cos(\theta - \tfrac{1}{2}\tan^{-1}(2a\cos(\theta)+r\cos(2\theta),\ 2(a+r\cos(\theta))\sin(\theta))) +
\end{aligned}$$

The following figures display these stress components curves for three selected directions θ. It's interesting to notice that, the provided solution satisfies to boundary conditions. In the immediate neighbourhood of the crack tip, stress components are infinites, moreover at infinity, $\frac{\sigma_{yy}}{\sigma_{\infty}}$ tends to 1 when the other components vanish, to satisfy to the remote stress distribution at infinity.
The provided expressions are general, depending both on the crack dimension (a), the loading magnitude (σ_{∞}) and can be evaluate at any point (r, θ) of the considered elastic region.

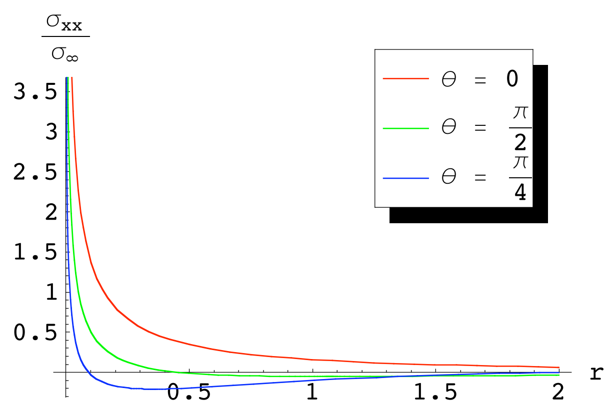


Figure 2.

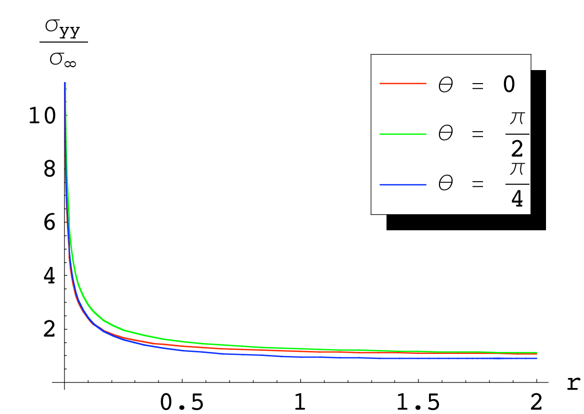
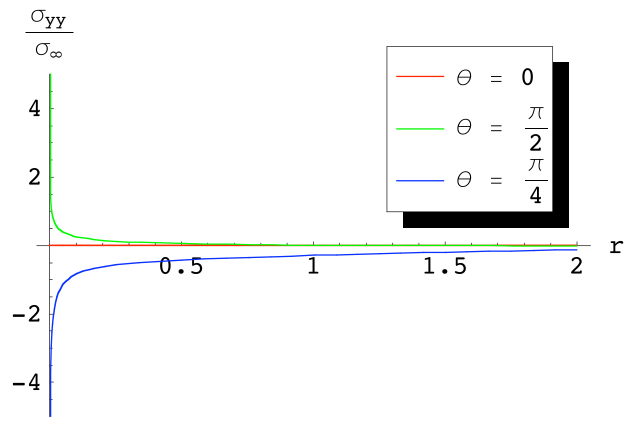


Figure 3.



$\frac{\sigma_{yy}}{\sigma_\infty}$
4
2
-2
-4
0.5
1
1.5
2
r
$\theta = 0$
$\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{4}$

Figure 4.

Following figures depicted stress components contourplots near the crack tip and reveal a stress distribution, which is in accordance with numerical predictions widely used in fracture mechanics.

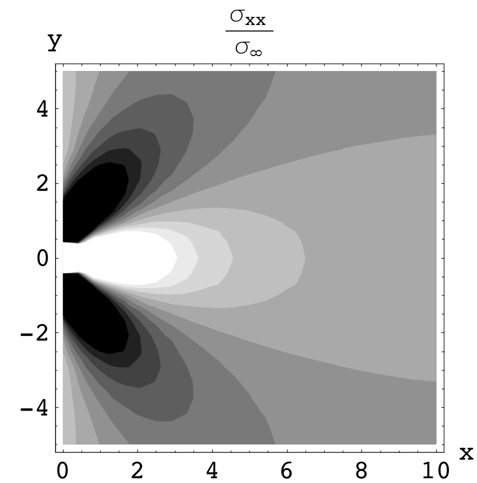


Figure 5.

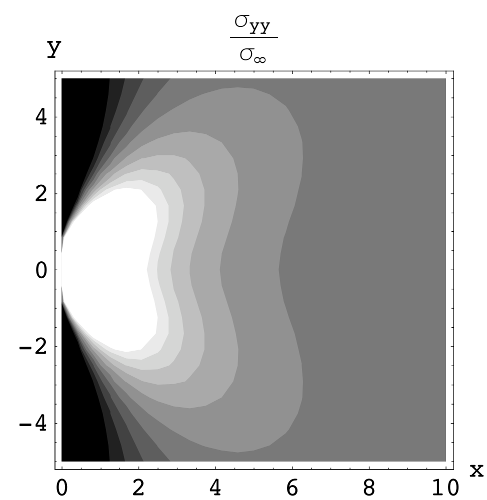


Figure 6.

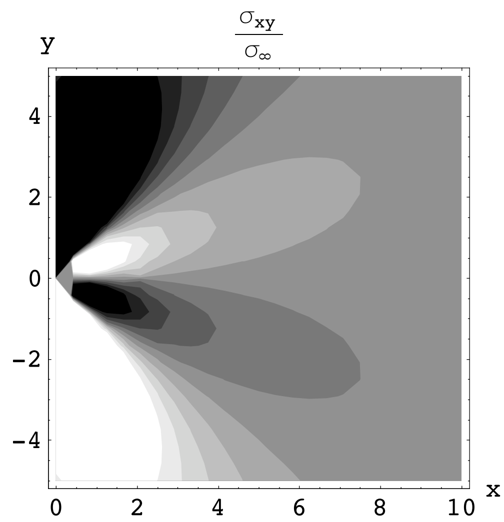


Figure 7.

Through the concept of complex functions, it's also possible to determine the expression of the stress intensity factor K generally used in fracture mechanics. Indeed, this stress intensity factor is related to the complex function ϕ:

$$K = K_I - iK_{II} = 2 \operatorname{Limit}_{z \longrightarrow z_0} \sqrt{2\pi (z - z_0)}\ \phi(z) \tag{14}$$

From relation (12 a), it's easy to obtain for the particulate treated case (crack under Mode I loading), the well known result: $K = K_I = \sigma_\infty \sqrt{\pi a}$.

This general routine can be easily applied to any specific boundary conditions as well at infinity as on the crack's contours in order to express stress field. Classical loadings such as Mode II and Mode III are also investigated and general analytical results obtained.

□ 2. Mode I for an infinite plate with two cracks

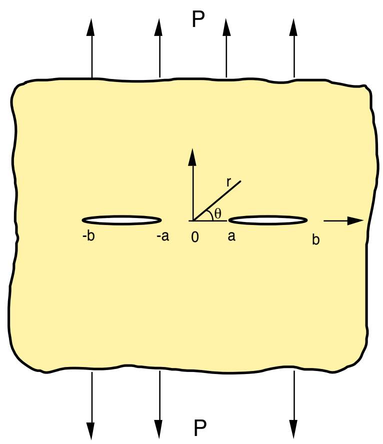


Figure 8.

Within the framework developed in the previous section, it's possible to obtain expressions of complex functions for this second configuration.

$$\phi = \frac{\frac{P z^2}{2} - \frac{b P E\left(\frac{b^2}{a^2}\right) a^2 - b P K\left(\frac{b^2}{a^2}\right) a^2 - b^2 P E\left(\frac{a^2}{b^2}\right) a + b^2 P K\left(\frac{a^2}{b^2}\right) a}{2\left(a K\left(\frac{a^2}{b^2}\right) - b K\left(\frac{b^2}{a^2}\right)\right)}}{\sqrt{(z^2-a^2)(z^2-b^2)}} - \frac{P}{4};$$

$$\Omega = \frac{P}{4} + \frac{\frac{P z^2}{2} - \frac{b P E\left(\frac{b^2}{a^2}\right) a^2 - b P K\left(\frac{b^2}{a^2}\right) a^2 - b^2 P E\left(\frac{a^2}{b^2}\right) a + b^2 P K\left(\frac{a^2}{b^2}\right) a}{2\left(a K\left(\frac{a^2}{b^2}\right) - b K\left(\frac{b^2}{a^2}\right)\right)}}{\sqrt{(z^2-a^2)(z^2-b^2)}}; \tag{15}$$

$E(m)$ gives the complete elliptic integral and $K(m)$ the complete elliptic integral of the first kind.

1. Stress intensity factor

The stress intensity factor at the crack tip is defined by equation (14). For the particular case of plate with two cracks this factor can be evaluated at the iner ($z_0 = \pm a$) or outer crack tip ($z_0 = \pm b$).

$$K_{Ia} = \frac{-P\sqrt{\pi}\, K\left(\frac{a^2}{b^2}\right) a^3 + b P\sqrt{\pi}\, E\left(\frac{b^2}{a^2}\right) a^2 + b^2 P\sqrt{\pi}\, K\left(\frac{a^2}{b^2}\right) a - a b^2 P\sqrt{\pi}\, E\left(\frac{a^2}{b^2}\right)}{\sqrt{a b^2 - a^3}\left(a K\left(\frac{a^2}{b^2}\right) - b K\left(\frac{b^2}{a^2}\right)\right)}$$

$$K_{Ib} = \frac{-P\sqrt{\pi}\, K\left(\frac{b^2}{a^2}\right) b^3 + a P\sqrt{\pi}\, E\left(\frac{a^2}{b^2}\right) b^2 - a^2 P\sqrt{\pi}\, E\left(\frac{b^2}{a^2}\right) b + a^2 P\sqrt{\pi}\, K\left(\frac{b^2}{a^2}\right) b}{\sqrt{b^3 - a^2 b}\left(a K\left(\frac{a^2}{b^2}\right) - b K\left(\frac{b^2}{a^2}\right)\right)} \tag{16}$$

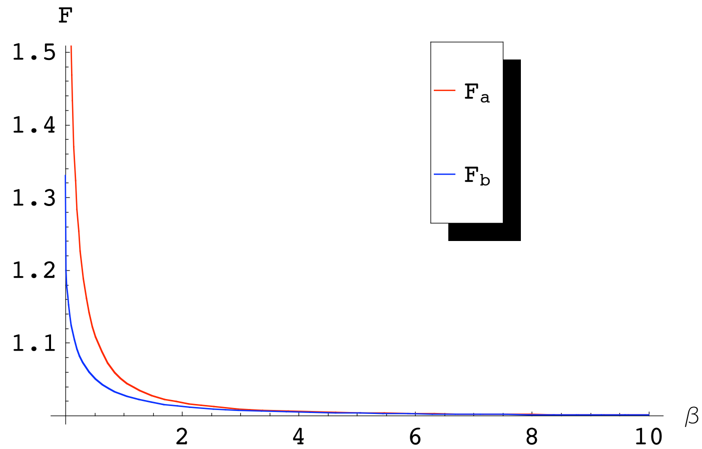
For further analysis, we introduce the following notation:
$K_I = F\ P\sqrt{\pi\,\frac{b-a}{2}}$ where $P\sqrt{\pi\,\frac{b-a}{2}}$ is the stress concentration factor for a single crack with $(b-a)$ length subjected to a mode I loading of magnitude P. The parameter $F \geq 1$, therefore represents the stress amplification coefficient due to interactions between cracks .

$$F_a = \frac{\sqrt{\frac{\beta}{\beta+1}}\left(\beta E\left(\frac{(\beta+2)^2}{\beta^2}\right)(\beta+2) - (\beta+2)^2 E\left(\frac{\beta^2}{(\beta+2)^2}\right) + 4(\beta+1) K\left(\frac{\beta^2}{(\beta+2)^2}\right)\right)}{2\beta K\left(\frac{\beta^2}{(\beta+2)^2}\right) - 2(\beta+2) K\left(\frac{(\beta+2)^2}{\beta^2}\right)};$$

$$F_b = \frac{(\beta+2)\left(-E\left(\frac{(\beta+2)^2}{\beta^2}\right)\beta^2 + (\beta+2) E\left(\frac{\beta^2}{(\beta+2)^2}\right)\beta - 4(\beta+1) K\left(\frac{(\beta+2)^2}{\beta^2}\right)\right)}{2\sqrt{\beta^2+3\beta+2}\left(\beta K\left(\frac{\beta^2}{(\beta+2)^2}\right) - (\beta+2) K\left(\frac{(\beta+2)^2}{\beta^2}\right)\right)} \tag{17}$$

Where the $\beta = \frac{\text{distance between cracks}}{\text{crack length}} = \frac{2\,a}{b-a}$.

Figure 9.



Cracks interactions coefficients are decreasing functions of the ratio β. It's also noticed that cracks interactions on the interior ends (F_a) are stronger than those in the external vicinity (F_b).
When the distance $2a$ between cracks is very weak relatively with their length $b-a$, interactions between the two cracks are important: higher than 1.2 and 1.5 respectively for inner and outer crack tips.
In the same way when the distance between the cracks becomes very large compared to the crack length, the interactions coefficients tend towards their minimal value 1.
One can conclude that for $\beta \geq 5$, interactions between the cracks become negligible.

In a previous investigation due to Erdogan and reported in [], the author suggested some expressions of these interactions coefficients at cracks tips. This semi-numerical approach is based on the following figure and provided the following results:

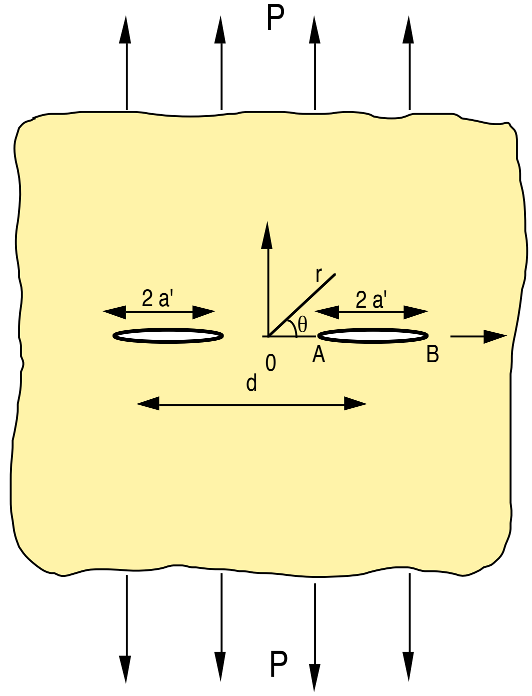


Figure 10.

$$
\begin{aligned}
F_{IB} &= \frac{d+2a'}{4a'}\left(\frac{d+2a'}{d}\right)^{\frac{1}{2}}\left(1-\frac{E(k)}{K(k)}\right);\\
F_{IA} &= \frac{d-2a'}{4a'}\left(\frac{d+2a'}{d}\right)^{\frac{1}{2}}\left(\left(\frac{d+2a'}{d-2a'}\right)^{2}\frac{E(k)}{K(k)}-1\right);\\
k &= \left(1-\left(\frac{d+2a'}{d-2a'}\right)^{2}\right)^{\frac{1}{2}};\\
K(k) &= \int_0^{\frac{\pi}{2}}\left(1-k^2\sin^2\theta\right)^{\frac{-1}{2}}\,d\theta;\\
E(k) &= \int_0^{\frac{\pi}{2}}\left(1-k^2\sin^2\theta\right)^{\frac{1}{2}}\,d\theta
\end{aligned}
\qquad (18)
$$

Following figures depict the interactions coefficients evolution as a function of the parameter β compared to the values predicted by the initial investigation due to Erdogan.

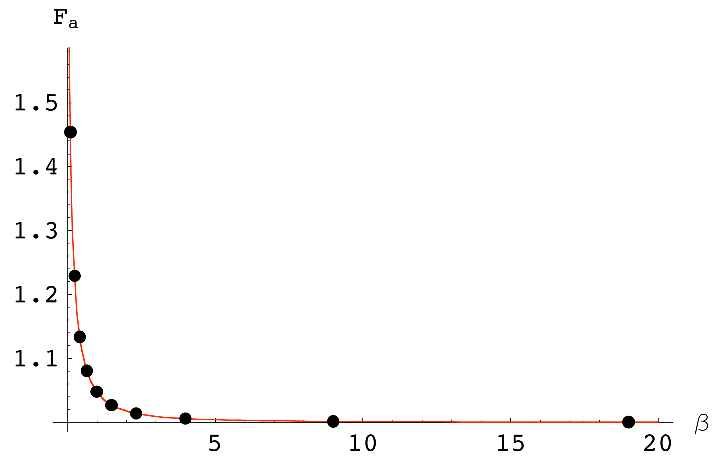


Figure 11.

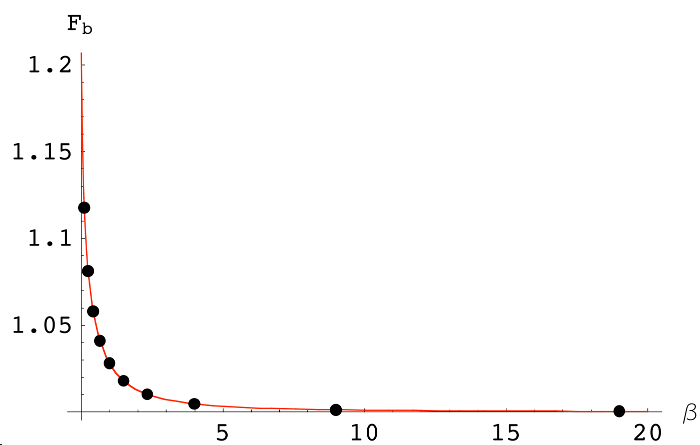


Figure 12.

Red curves are relative to our proposed results where plotted points are obtained form Erdogan data. Results provided by the present study match very well these initial semi-numerical data of Erdogan. Moreover, it's also possible to obtain full analytical expressions of stress fields in the considered elastic plate.

■ 3. Conclusion

This present paper reports a study on the determination of the analytical expressions of the stress fields and the stress intensity factor K in an elastic region containing cracks.
Expressions of these parameters classically used in fracture mechanics are available but they appear to be crude approximations of our results which are applicable beyond the crack vicinity. Our expressions have thus the merit to be more general than those resulting from other analytical methods.
Thus the results obtained by Irwin by application of the method of Westergaard are approached expressions, only valid for $r \ll 1$. These results constitute a good approximation of the stress fields and displacements in the vicinity of the crack.
For the problem of the elastic region with two cracks, expressions of stress field are provided. Their use by computer algebra software such as Mathematica does not present a major difficulty. They can also be integrated in a optimized form into a numerical computer code.
For engineers concerned with design and technical calculation on the safety of structures and thus always willing to gain a more precise fracture criterion, results provided by the present study are a real progress.

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