Convergence improvement of infinite series by linear fractions

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■ Abstract

Though the Gregory–Leibniz series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ converges slowly to $\frac{\pi}{4}$, a linear (red) fraction of n accelerates the speed of convergence as follows;

In[3]:=
$$\text{Table}\left[N\left[\left\{4\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1},\ (-1)^{n}\frac{1}{n}+4\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1}\right\}, 20\right], \{n, 1000, 1005\}\right] \text{ // TableForm}$$

Out[3]//TableForm=

3.1405926538397929260	3.1415926538397929260
3.1425916543395430509	3.1415926533405420519
3.1405946498462829411	3.1415926538382989091
3.1425896623151108712	3.1415926533420301135
3.1405966379005617930	3.1415926538368167731
3.1425876782191282440	3.1415926533435063534

Joseph Roy North observed an analogous phenomenon with respect to the truncated value of the series in 1988. In the present article we explain the essence of the phenomenon in an elementary and general method, and generate linear fraction terms in order to accelerate the convergence speed for various infinite series by virtue of *Mathematica* commands. For example the following acceleration by linear fractions for

$\zeta(2)$ almost equals L.Euler's ingenious acceleration $(\log 2)^2 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1} n^2}$ in his great analysis on Basel problem.

In[14]:=
$$f[n\_] := \frac{1}{60}\left(\frac{1}{n-2} + \frac{1}{n+3}\right) - \frac{2}{15}\left(\frac{1}{n-1} + \frac{1}{n+2}\right) + \frac{37}{60}\left(\frac{1}{n} + \frac{1}{n+1}\right) + \sum_{k=1}^{n} \frac{1}{k^2};$$

$$N\left[\left\{f[20], \text{Zeta}[2], \text{Log}[2]^2 + \sum_{n=1}^{20} \frac{1}{2^{n-1} n^2}\right\}, 12\right] //$$

TableForm

Out[15]//TableForm=
```
1.64493407030
1.64493406685
1.64493406287
```

The following is another example of acceleration by linear fraction terms. A few fraction terms make the approximation accurate.

In[23]:=
$$g[n\_] := \frac{1}{180}\left(\frac{1}{n-1} - \frac{1}{n+2}\right) + \frac{11}{60}\left(\frac{1}{n+1} - \frac{1}{n}\right) + \sum_{k=1}^{n} \frac{1}{k} - \frac{\text{Log}[n] + \text{Log}[n+1]}{2};$$

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N[{g[20], EulerGamma}, 12] // TableForm
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Out[23]//TableForm=
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0.577215665977
0.577215664902
```

■ Goldbach's method

In 1729 Goldbach had an inequality estimation for $\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2}$,

$$\sum_{k=7}^{\infty} \frac{1}{\left(k - \frac{7}{16}\right)\left(k + \frac{9}{16}\right)} + \sum_{k=1}^{6} \frac{1}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2} < \sum_{k=7}^{\infty} \frac{1}{\left(k - \frac{1}{2}\right)\left(k + \frac{1}{2}\right)} + \sum_{k=1}^{6} \frac{1}{k^2}, \quad (1)$$

that is,

$$\frac{41423}{25200} < \sum_{k=1}^{\infty} \frac{1}{k^2} < \frac{76997}{46800}. \quad (2)$$

Applying his method to more general situation, we can improve the convergence of infinite series. That is , if the n–th partial sum of $\sum_{k=1}^{\infty} a_k$ is revised by the term $\sum_{k=n+1}^{\infty} b_k$, the convergence speed of a sequence $\{\sum_{k=n+1}^{\infty} b_k + \sum_{k=1}^{n} a_k\}$ is equal to that of $\{\sum_{k=1}^{n} (a_k - b_k)\}$.

$$\sum_{k=n+1}^{\infty} b_k + \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} b_k + \sum_{k=1}^{n} (a_k - b_k) \quad (3)$$

In the case of $a_k = \frac{1}{k^2}$, $b_k = \frac{1}{k^2 - \frac{1}{4}}$ for the above, we observe that the linear fraction of n,

In[11]:= $\sum_{k=n+1}^{\infty} \frac{1}{k^2 - \frac{1}{4}}$

Out[11]= $\frac{2}{1+2\,n}$

improves the convergence speed of $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2}$.

In[26]:= Table$\left[N\left[\left\{\frac{2}{1+2\,n} + \sum_{k=1}^{n} \frac{1}{k^2}, \text{Zeta}[2]\right\}, 15\right], \{n, 30, 40\}\right]$ //

TableForm

Out[26]//TableForm=

1.64493700284750	1.64493406684823
1.64493673207396	1.64493406684823
1.64493649359716	1.64493406684823
1.64493628274213	1.64493406684823
1.64493609562296	1.64493406684823
1.64493592899170	1.64493406684823
1.64493578011944	1.64493406684823
1.64493564670206	1.64493406684823
1.64493552678504	1.64493406684823
1.64493541870314	1.64493406684823
1.64493532103163	1.64493406684823

We give another example of acceleration of the convergence speed of $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2}$.

In[1]:= $$\text{Series}\left[\frac{1}{n^2}-\left(\frac{p}{(n-1)\,(n+1)}+\frac{q}{(n-2)\,(n+2)}+\frac{r}{(n-3)\,(n+3)}\right),\ \{n,\ \infty,\ 8\}\right]$$

Out[1]= $$(1-p-q-r)\left(\frac{1}{n}\right)^2+(-p-4q-9r)\left(\frac{1}{n}\right)^4+(-p-16q-81r)\left(\frac{1}{n}\right)^6+(-p-64q-729r)\left(\frac{1}{n}\right)^8+O\left[\frac{1}{n}\right]^9$$

In[2]:= $$\text{values}=\text{Solve}[\{1-p-q-r==0,\ -p-4q-9r==0,\ -p-16q-81r==0\},\ \{p,\ q,\ r\}]$$

Out[2]= $$\left\{\left\{p\to\frac{3}{2},\ q\to-\frac{3}{5},\ r\to\frac{1}{10}\right\}\right\}$$

The finite sum of linear fractions,

In[3]:= $$\sum_{k=n+1}^{\infty}\left(\frac{p}{(k-1)\,(k+1)}+\frac{q}{(k-2)\,(k+2)}+\frac{r}{(k-3)\,(k+3)}\right)\ /.\ \text{values}[[1]]\ //\ \text{Apart}$$

Out[3]= $$\frac{1}{60\,(-2+n)}-\frac{2}{15\,(-1+n)}+\frac{37}{60\,n}+\frac{37}{60\,(1+n)}-\frac{2}{15\,(2+n)}+\frac{1}{60\,(3+n)}$$

improves the convergence speed of $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}$ much more.

In[1]:= $$\text{Table}\left[N\left[\left\{\frac{1}{60}\left(\frac{1}{n+3}+\frac{1}{n-2}\right)-\frac{2}{15}\left(\frac{1}{n+2}+\frac{1}{n-1}\right)+\frac{37}{60}\left(\frac{1}{n}+\frac{1}{n+1}\right)+\sum_{k=1}^{n}\frac{1}{k^2},\ \text{Zeta}[2]\right\},\ 15\right],\ \{n,\ 30,\ 40\}\right]\ //\ \text{TableForm}$$

Out[1]//TableForm=

1.64493406705963	1.64493406684823
1.64493406701680	1.64493406684823
1.64493406698360	1.64493406684823
1.64493406695767	1.64493406684823
1.64493406693727	1.64493406684823
1.64493406692110	1.64493406684823
1.64493406690820	1.64493406684823
1.64493406689784	1.64493406684823
1.64493406688948	1.64493406684823
1.64493406688269	1.64493406684823
1.64493406687715	1.64493406684823

■ EulerGamma

For Euler's constant $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)$ we utilize the following equality.

In[57]:= $\text{FullSimplify}\left[\sum_{k=1}^{n}\frac{1}{k}-\frac{\text{Log}[n]+\text{Log}[n+1]}{2}==1-\frac{1}{2}\text{Log}[2]+\sum_{k=2}^{n}\left(\frac{1}{k}+\frac{1}{2}\text{Log}\left[\frac{1-\frac{1}{k}}{1+\frac{1}{k}}\right]\right),\ \text{Assumptions}\to\{n>0,\ n\in\text{Integers}\}\right]$

Out[57]= True

We can find linear fraction terms for the acceleration of convergence by the same method.

In[58]:= $\text{Series}\left[\frac{1}{k}+\frac{1}{2}\text{Log}\left[\frac{1-\frac{1}{k}}{1+\frac{1}{k}}\right]-\left(\frac{s}{(k-1)\,k\,(k+1)}+\frac{t}{(k-2)\,k\,(k+2)}\right),\ \{k,\infty,8\}\right]$

Out[58]= $\left(-\frac{1}{3}-s-t\right)\left(\frac{1}{k}\right)^3+\left(-\frac{1}{5}-s-4\,t\right)\left(\frac{1}{k}\right)^5+\left(-\frac{1}{7}-s-16\,t\right)\left(\frac{1}{k}\right)^7+O\left[\frac{1}{k}\right]^9$

In[59]:= $\text{val}=\text{Solve}\left[\left\{-\frac{1}{3}-s-t==0,\ -\frac{1}{5}-s-4\,t==0\right\},\ \{s,t\}\right]$

Out[59]= $\left\{\left\{s\to-\frac{17}{45},\ t\to\frac{2}{45}\right\}\right\}$

In[60]:= $\sum_{k=n+1}^{\infty}\left(\frac{s}{(k-1)\,k\,(k+1)}+\frac{t}{(k-2)\,k\,(k+2)}\right)$ /. val[[1]] // Apart

Out[60]= $\frac{1}{180\,(-1+n)}-\frac{11}{60\,n}+\frac{11}{60\,(1+n)}-\frac{1}{180\,(2+n)}$

Reference

[1] Andre Weil, Number theory, Birkhauser, 1983.
[2] Jonathan Borwein and David Bailey, Mathematics by Experiment, A K Peters, 2004.
[3] Nico M. Temme, Special functions, John Wiley & Sons, Inc., 1996.