An isothermal gas flow in porous medium

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This paper is concerned with a transient flow of gas within a two−dimensional porous medium.

The porous medium is filled with gas under uniform pressure (the initial condition). The edges of considered porous sample are insulated, except one section of edge which is opened (the boundary conditions). Pressure outside the sample is lower than pressure in the porous medium. Therefore the outflow of gas observed. The phenomena is an initial−boundary value problem. Method based on Picard iteration method and one of meshless methods is proposed to solve considered problem. Numerical implementation of the method is included in this work. Results of numerical experiment are illustrated graphically and disscussed.

■ Nomenclature

φ – porosity
μ – viscosity [Pa s]
k – permability [darcys, m^2]
ρ – mass density of the fluid [kg/m^3]
R – individual gas constant [J/kg /K]
g – gravity acceleration [m/s^2]
p – pressure [Pa]
q – superficial fluid velocity [m/s]
T – temperature [K]
t – time [s]
x, y – geometry variables [m]
a, b, c – geometry parameters [m]
p_0 – inital gas pressure in reservoir [Pa]
p_1 – gas pressure outside the reservoir [Pa]
τ – dimensionless time parameter
X, Y – dimensionless geometry variables
D, E – dimensioless geometry parameters
P – dimensioless pressure
P_1 – dimensioless gas pressure outside the reservoir
$P^{(n,i)}$ – dimensionless pressure at i–th iteration within n–th time step

■ Introduction

A problem of isothermal gas flow has been widely disscussed in literature. In recent years many methods to solve such problem has been proposed and developed. During few last decades more and more popular became meshless methods. One those methods is Trefftz method based on fundamental solutions or T−Herrera functions. Originally, method was used for solving homogeneous linear problems.

But the usage of the method has been developed. Examples of solving boundary value problem with linear equation with non−linear boundary conditions by Trefftz method are given in papers [1−2]. Second case of boundary value problem solved by the method is problem defined by non−linear Poisson equation (see [3−7]).

$$\nabla^2 u = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \qquad (1)$$

where u is unknown function, and f is known function in which some arguments are unknown.

In paper [3] the non−linear thermal explosions problem was solved by method of fundamental solutions. The radial basis functions were used for interpolation of right hand side function, Picard iteration method was used to treat non−linearity. In paper [4] method called as 'particular solution Trefftz method' was used. It is an extension and improvement of ideas proposed in paper [3]. Another version of Trefftz method for solution of non−linear Poisson equation was presented in paper [5]. For non−linear thermal conductivity problem by Kirchoff transformacion the non−linearity exists only in boundary conditions. The non−linear algebraic equation was solved by stabilized continuation method. Kita at. al [6] considered steady state heat conduction problems for functionally gradient materials. For overcoming the difficulty with non−linear Poisson equation authors presented the combination scheme of the Trefftz method with the bomputiong point analysis method. Also steady state heat conduction problem with temperature dependent conductivity was considered in paper [7]. Combination of fundamental solutions method with Picard iteration was used for non−linear Poisson equation. Evolutionary algorithm was applied for optimal determination of method parameters. More complicated application of Trefftz method was presented in paper [8] where the method of operator splitting with method of fundamental solution for transint non−linear Poisson problems. These problems are widely encountered in the modelling many physical phenomena. Governing differential equation has a form

$$\frac{\partial u}{\partial t} = \nabla^2 u + f(u) \qquad (2)$$

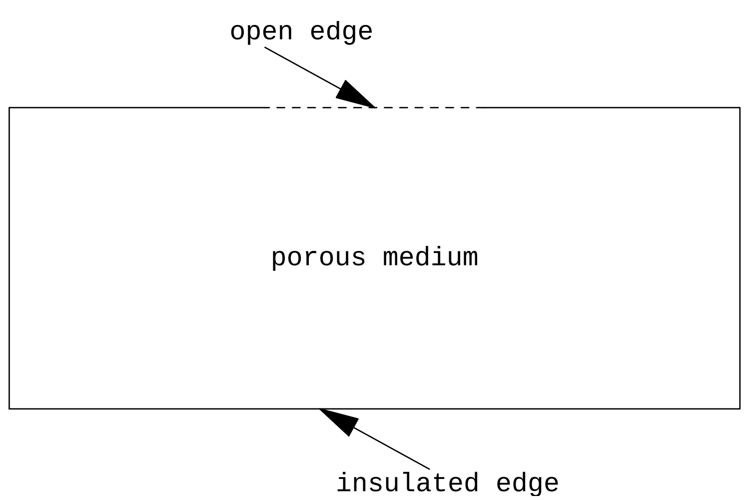
where t is time.

Authors of [9] proposed method to solve porblem (2) by a Trefftz type method. One of the method they suggested is finite differencing in time, which transform equation (2) to a sequence of coupled stationary equations.

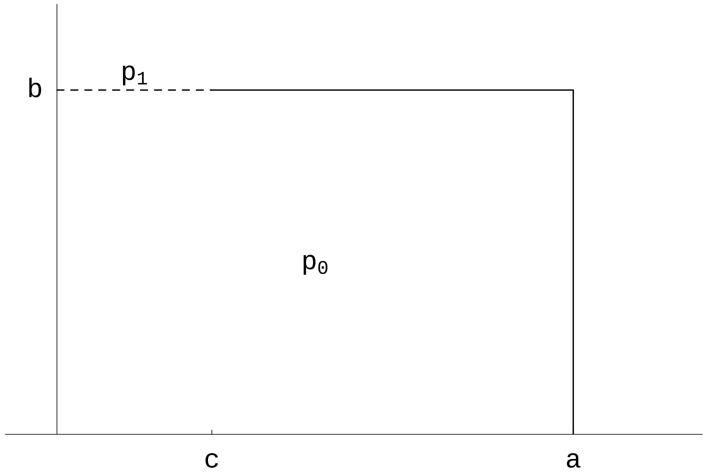
The purpoes of the present paper is application of some kind of Trefftz method to a problem of the transint flow of gas within a two–dimensional porous medium. Unsteady gas flow through semi–infinite porous medium was considered in paper [10]. In such case problem is described by ordinary differential equation. In case of finite porous region the governing equation for pressure of gas is a partial differential equation with two independent geometrical variables and time variable. In our proposition the method of fundamental solution for spatial variables and finite difference method for time variable are employed to obtain a solution of the non–linear partial differential equation describing the flow of gas. The inhomogeneous term is expressed by radial basis functions at each time step. Picard iteration is used for treating nonlinearity.

■ Problem description

Considered region of the porous medium with flowing fluid is presented on Figure 1. The porous medium is filled with gas under uniform pressure. The edges of considered reserwoir are insulated, except one piece of edge which is opened. Pressure outside the reserwoir is lower than pressure in porous medium.



The reservoir is rectangle with edge of length 2a and b. The open edge has length equal to 2c. Because of symmetry of geometry and phenomena, the half of region will be considered. Figure 2 presents the geometry of the region taken into account.



For investigation of gas flow in porous medium we introduce following assumptions:

– flow of gas follows Darcy's law

– only phase flowing is a gas of constant composition and viscosity

– gas is perfect and gas flow is isothermal

– permeability of the porous medium is constant and uniform

– gravitational forces are neglected.

■ Motion equations

Darcy's Law is filtration equation for fluid flow in porous medium and in 2–D case has form

$$q_x = -\frac{k}{\mu}\frac{\partial p}{\partial x} \tag{3}$$

$$q_y = -\frac{k}{\mu}\frac{\partial p}{\partial y} \tag{4}$$

Continuity equation for porous media is

$$\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) = -\frac{\partial}{\partial t}(\varphi\rho) \tag{5}$$

The gas equation for isothermal phenomena will be used, as well:

$$\rho = \frac{p}{RT} \tag{6}$$

where T – temperature is constant.

Applying the qe.(4) and eq. (5) to the eq. (6) gives

$$\frac{\partial}{\partial x}\left(\frac{p}{RT}\frac{k}{\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{p}{RT}\frac{k}{\mu}\frac{\partial p}{\partial y}\right) = \frac{\mu}{k}\frac{\partial}{\partial t}\left(\varphi\frac{p}{RT}\right) \tag{7}$$

Rearranging the eq. (7) yields to equation

$$\frac{\partial}{\partial x}\left(p\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial p}{\partial y}\right) = \frac{\mu}{k}\frac{\partial}{\partial t}(\varphi p) \tag{8}$$

which describes the unstedy isothermal flow of gas in porous medium.

The solution of equation (8) depends on initial and boudary condtions. They are defined for considered region, shown by Figure 2.

The initial condition says that there is uniform pressure in porous medium:

$$p(x, y, t) = p_0 \tag{9}$$

for $t=0$, $0<x<a$ and $0<y<b$.

The boundary condition at open edge $\{(x,y)|(0<x<c)\cap(y=b)\}$ is

$$p(x, y, t) = p_1 < p_0 \tag{10}$$

for $0<t<\infty$.

For insulated edges $\{(x,y)|((0<x<a)\cap(y=0))\cup((c<x<a)\cap(y=b))\cup((x=a)\cap(0<y<b))\}$ boundary condition is

$$\frac{\partial p}{\partial n} = 0 \tag{11}$$

and for $\{(x,y)|(x=0)\cap(0<y<b)\}$ the symmetry condtion is applied

$$\frac{\partial p}{\partial n} = 0 \tag{12}$$

for $0<t<\infty$.

the dimantionless variables are introduced

$$X = \frac{x}{a} \quad Y = \frac{y}{a} \quad E = \frac{b}{a} \quad D = \frac{c}{a} \quad P = \frac{p}{p_0} \quad P_1 = \frac{p_1}{p_0} \quad \tau = \frac{k p_0}{\varphi}\mu a^2 t \tag{13}$$

Therefore, the equation (8) has a dimensionless form

$$\frac{\partial}{\partial x}\left(P\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(P\frac{\partial P}{\partial y}\right) = \frac{\partial P}{\partial \tau} \tag{14}$$

and the initial condition is

$$P(X, Y, \tau) = 1 \quad (15)$$

for $\tau=0$, $0<X<1$, $0<Y<E$.

The boundary conditions in dimensionless form are:

the boundary condtion for the open edge

$$P(X, Y, \tau) = P_1 < 1 \quad (16)$$

for $\tau=0$, $0<X<D$, $Y=E$.

and the inslutation and symmetry condition

$$\frac{\partial P}{\partial n} = 0 \quad (17)$$

for the boundary $\{(X,Y)|((0<X<1)\cap(Y=0))\cup((D<X<1)\cap(Y=E))\cup((X=1)\cap(0<Y<E))\}$.

■ Algorithm for solving initial−boundary problem

Assming that time derivative term cam be expanded using finite difference

$$\frac{\partial P}{\partial \tau} = \frac{P^{(n+1)} - P^{(n)}}{\Delta\tau} \quad (18)$$

for $n=0,1,2,...$ equation (14) can be approximated as

$$\frac{\partial^2 P^{(n+1)}}{\partial X^2} + \frac{\partial^2 P^{(n+1)}}{\partial Y^2} - \frac{P^{(n+1)} - P^{(n)}}{P^{(n)}\,\Delta\tau} = -\frac{1}{P^{(n)}}\left\{\left(\frac{\partial P^{(n)}}{\partial X}\right)^2 + \left(\frac{\partial P^{(n)}}{\partial Y}\right)^2\right\} \quad (19)$$

with the initial condtion

$$P^{(0)}(X, Y, \tau) = 1 \quad (20)$$

for $\tau=0$, $0<X<1$, $0<Y<E$.

and the boundary conditions

$$P^{(n+1)} = P_1^{(n+1)} < 1 \quad (21)$$

for $\tau=0$, $0<X<D$, $Y=E$
and

$$\frac{\partial P^{(n+1)}}{\partial n} = 0 \quad (22)$$

for the boundary
$\{(X,Y)|((0<X<1)\cap(Y=0))\cup((D<X<1)\cap(Y=E))\cup((X=0)\cap(0<Y<E))\cup((X=1)\cap(0<Y<E))\}$.

where $P^{(n)}$ is dimensionless pressure at n–th time step, $P^{(n+1)}$ is dimensionless pressure at (n+1)–th time step.
For the first step the dimensionless pressure P is uniform. Therefore the equation (19) may be treated as the Helmholtz equation:

$$\frac{\partial^2 P^{(1)}}{\partial X^2} + \frac{\partial^2 P^{(1)}}{\partial Y^2} - k^2 P^{(1)} = f(X, Y) \tag{23}$$

where $k^2 = 1/(P^{(0)}\,\Delta\tau)$, $f(X,Y) = -1/(\Delta\tau)$.
The boundary condtions for the equation (23) are

$$P^{(1)} = P_1 < 1 \tag{24}$$

for $0<X<D$, $Y=E$
and

$$\frac{\partial P^{(1)}}{\partial n} = 0 \tag{25}$$

for the boundary
$\{(X,Y)\,|\,((0<X<1)\cap(Y=0))\cup((D<X<1)\cap(Y=E))\cup((X=0)\cap(0<Y<E))\cup((X=1)\cap(0<Y<E))\}$.

The calculation of pressure in the next time steps is based also on eq. (19). However the pressure distribution is not uniform anymore (as the result from the first time step), the equation is tranformed into Poisson equation:

$$\frac{\partial^2 P^{(n+1)}}{\partial X^2} + \frac{\partial^2 P^{(n+1)}}{\partial Y^2} = \frac{1}{\Delta\tau} - \frac{P^{(n)}}{P^{(n+1)}\,\Delta\tau} - \frac{1}{P^{(n+1)}}\left\{\left(\frac{\partial P^{(n+1)}}{\partial X}\right)^2 + \left(\frac{\partial P^{(n+1)}}{\partial Y}\right)^2\right\} \tag{26}$$

with boundary conditions (24, 25). The equation is strongly non–linear with respect to $P^{(n+1)}$, therefore, it is solved in an iterative fashion:

$$\frac{\partial^2 P^{(n+1,i+1)}}{\partial X^2} + \frac{\partial^2 P^{(n+1,i+1)}}{\partial Y^2} = \frac{1}{\Delta\tau} - \frac{P^{(n)}}{P^{(n+1,i)}\,\Delta\tau} - \frac{1}{P^{(n+1,i)}}\left\{\left(\frac{\partial P^{(n+1,i)}}{\partial X}\right)^2 + \left(\frac{\partial P^{(n+1,i)}}{\partial Y}\right)^2\right\} \tag{27}$$

with boundary conditions (25, 26), where $P^{(n+1,i)}$ is the i–th iteration result at $(n+1)$–th time step. We introduce an initial condition for interative procedure e.q. trial equation in Laplace form, which is modified version of eq. (27):

$$\frac{\partial^2 P^{(n+1,1)}}{\partial X^2} + \frac{\partial^2 P^{(n+1,1)}}{\partial Y^2} = 0 \tag{28}$$

with boundary conditions

$$P^{(n+1,1)} = P_1 < 1 \tag{29}$$

for $0<X<D$, $Y=E$
and

$$\frac{\partial P^{(n+1,1)}}{\partial n} = 0 \tag{30}$$

for the boundary
$\{(X,Y)|((D<X<1)\cap(Y=E))\cup((X=0)\cap(0<Y<E))\cup((X=1)\cap(0<Y<E))\}$.

One extra boundary condtion is added

$$P^{(n+1,1)} = P^{(n)} \tag{31}$$

for $\{(X,Y)|(0<X<1)\cap(Y=0)\}$.

to combine the previous time step pressure distribution with the solution at the next time step. Equation (28) is solved by the fundamental solution method, including the appropriate boundary conditions into calculation.
Solution at second and next iteration steps is found by Trefftz method, based on the eq. (27) with its boundary conditions. Therefore, in one time step we obtain the sequence of solutions:
$P^{(n+1,1)}, P^{(n+1,2)}, \ldots$.
The iterative proces is terminated when difference between solutions of two successive iteration steps is quite small, less than a chosen small parameter. We introduce m, which points the iteration step number, at which solution is taken as the solution at n–th time step, noticed as
$P^{(n+1,m)} = P^{(n+1)}$.

■ Trefftz method to solve boundary problem

Partial differential inhomogeneous equation

$$Lu = f(x, y) \tag{32}$$

is considered on the region Ω.

Operator L is a partial differential operator, which includes Laplace operator.

The boundary condition has the general form

$$Bu = g(x, y) \tag{33}$$

where B is an operator imposed as boundary conditions, such Dirichlet, Neumann and Robin. Let us denote $\{(x_i, y_i)\}_{i=1}^{N}$ to be N collocation points in $\Omega \cup \partial\Omega$ of which $\{(x_i, y_i)\}_{i=1}^{Nl}$ are interionr points; $\{(x_i, y_i)\}_{i=N1+1}^{N}$ are boundary points.
The right–hand side function f is approximated by Radial Basis Functions (RBFs) as

$$f_N(x, y) = \sum_{j=1}^{N} a_j \varphi(r_j) + \sum_{k=1}^{l} b_k p_k(x, y) \tag{34}$$

where $r_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}$ and $\varphi(r_j) : R^d \to R^+$ is a RBF, $\{p_k\}_{k=1}^{l}$ is the complete basis for d−variate polynomials of degree $\leq m-1$, and C_{m+d-1}^{d} is the dimensions of P_{m-1}. The coefficients $\{a_j\}$, $\{b_k\}$ can be found by solving the system

$$\sum_{j=1}^{N} a_j \varphi(r_{ji}) + \sum_{k=1}^{l} b_k p_k(x_i, y_i) = f_N(x_i, y_i) \text{ for } 1 \leq i \leq N \tag{35}$$

$$\sum_{k=1}^{l} a_j p_k(x_j, y_j) = 0 \qquad \text{for } 1 \leq k \leq l \tag{36}$$

where $r_{ji} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$, $\{(x_i, y_i)\}_{i=1}^{L}$ are the collocation points on $\Omega \bigcup \partial\Omega$.

The approximate particular solution u_p can be obtained using the coefficients $\{a_j\}$ and $\{b_k\}$ by

$$u_p = \sum_{j=1}^{N} a_j \phi(r_j) + \sum_{k=1}^{l} b_k \psi_k(x, y) \tag{37}$$

where

$$L\phi = \varphi \tag{38}$$

$$L\psi_k = p_k \tag{39}$$

Solution of differential euqation (32) now can be given as

$$u = u_p + v \tag{40}$$

where v is solution of boundary value problem in the form

$$Lv = 0 \qquad \text{in } \Omega \tag{41}$$

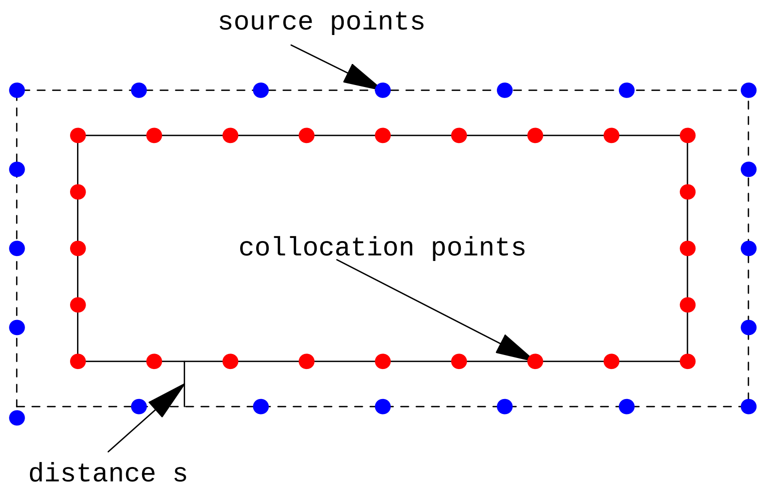
$$Bv = g(x, y) - Bu_p \qquad \text{on } \Omega \tag{42}$$

The method of fundamental solution is used to solve problem presented above, what means that

$$v = \sum_{j=1}^{N} c_j f_S(r_j) \tag{43}$$

where $f_S(r_j)$ is the fundamental solution function.

To avoid singularity of fundamental solution function a set of source points is introduced. The source points are chosen outside the considered region, with distance s from the region boundary (see Figure 3).



Putting (43) into boundary condition (42)

$$\sum_{j=1}^{N} c_j \, Bf_S \, (r_{ji}) = g\,(x_i, y_i) - Bu_p\,(x_i, y_i) \qquad \text{for } 1 \le i \le N \quad (44)$$

gives the system of linear equations, which solution is the set of c_j . The solution of the boundary problem (32) and (33) is calculated by equation (40).

□ Numerical implementation

For Helmholtz equation, the differential operator is

$$L = \nabla^2 - k^2 \qquad (45)$$

where ∇^2 is the Laplace operator and k is constant real number.
The inhomogeneous boundary problem (32) with operator (45) is solved by numeriacl implementation of the solution given by equation (37), (43) and (41). The function f is approximated as is presented in formula (34). The radial basis functions are [11]:

Case 1

$$\varphi\,(r) = \begin{cases} 0 \text{ for } r = 0 \\ r^2 \ln r \text{ for } r \neq 0 \end{cases} \qquad (46)$$

Case 2

$$\varphi\,(r) = \begin{cases} 0 \text{ for } r = 0 \\ r^4 \ln r \text{ for } r \neq 0 \end{cases} \qquad (47)$$

Case 3

$$\varphi\,(r) = \begin{cases} 0 \text{ for } r = 0 \\ r^6 \ln r \text{ for } r \neq 0 \end{cases} \qquad (48)$$

Case 4

$$\varphi\,(r) = \begin{cases} 0 \text{ for } r = 0 \\ r^8 \ln r \text{ for } r \neq 0 \end{cases} \qquad (49)$$

Case 5

$$\varphi(r) = \begin{cases} 0 \text{ for } r = 0 \\ r^{10} \ln r \text{ for } r \neq 0 \end{cases} \tag{50}$$

The solutions of the problem (39) with right−hand side function given by (46), (47), (48), (49) and (50) are, appropriate:

Case 1

$$\phi(r) = \begin{cases} \frac{4}{k^4}\left(\gamma + \ln\frac{k}{2} - 1\right) \text{ for } r = 0 \\ -\frac{4}{k^4}\left(K_0(kr) + \ln r + 1\right) - \frac{r^2 \ln r}{k^2} \text{ for } r > 0 \end{cases} \tag{51}$$

Case 2

$$\phi(r) = \begin{cases} \frac{64}{k^6}\left(\gamma + \ln\frac{k}{2}\right) - \frac{96}{k^6} \text{ for } r = 0 \\ -\frac{64}{k^6}\left(K_0(kr) + \ln r\right) - \frac{r^2 \ln r}{k^2}\left(\frac{16}{k^2} + r^2\right) - \frac{8 r^2}{k^4} - \frac{96}{k^6} \end{cases} \tag{52}$$

Case 3

$$\phi(r) = \begin{cases} \frac{2304}{k^8}\left(\gamma + \ln\frac{k}{2}\right) - \frac{4224}{k^8} \text{ for } r = 0 \\ -\frac{2304}{k^8}\left(K_0(kr) + \ln r\right) - \frac{r^2 \ln r}{k^2}\left(\frac{576}{k^4} + \frac{36 r^2}{k^2} + r^4\right) \\ -\frac{12 r^2}{k^4}\left(\frac{40}{k^2} + r^2\right) - \frac{4224}{k^8} \end{cases} \tag{53}$$

Case 4

$$\phi(r) = \begin{cases} \frac{147456}{k^{10}}\left(\gamma + \ln\frac{k}{2}\right) - \frac{307200}{k^{10}} \text{ for} \\ -\frac{147456}{k^{10}}\left(K_0(kr) + \ln r\right) - \frac{r^2 \ln r}{k^2}\left(\frac{36864}{k^6} + \frac{2304 r^2}{k^4}\right. \\ -\frac{r^2}{k^4}\left(\frac{39936}{k^4} + \frac{1344 r^2}{k^2} + 16 r^4\right) - \frac{307200}{k^{10}} \end{cases} \tag{54}$$

Case 5

$$\phi(r) = \begin{cases} \frac{14745600}{k^{12}}\left(\gamma + \ln\frac{k}{2}\right) - \frac{33669120}{k^{12}} \text{ for r} \\ -\frac{14745600}{k^{12}}\left(K_0(kr) + \ln r\right) \\ -\frac{r^2 \ln r}{k^2}\left(\frac{3686400}{k^8} + \frac{230400 r^2}{k^6} + \frac{6400 r^4}{k^4} + \frac{6100 r^6}{k^2}\right. \\ -\frac{r^2}{k^4}\left(\frac{4730880}{k^6} + \frac{180480 r^2}{k^4} + \frac{2880 r^4}{k^2} + 20 r^6\right) - 336 \end{cases} \tag{55}$$

where $K_0(r)$ is Bessel function of the second kind.

The fundamental solution for Helmholtz equation with operator (45) is the Bessel function:

$$f_S(r) = K_0(kr) \tag{56}$$

Solution of Poisson inhomogeneous equation is presented in [12, 13]. However, some radial basis functions and solutions of Poisson equations are included in this paper for reader convenience.

In Poisson equation differential operator is the Laplace operator:

$$L = \nabla^2 \tag{57}$$

The radial basis functions are:

Case 1

$$\varphi(r) = r^k \tag{58}$$

Case 2

$$\varphi(r) = \begin{cases} 0 \text{ for } r = 0 \\ r^k \ln r \text{ for } r \neq 0 \end{cases} \tag{59}$$

Case 3

$$\varphi(r) = \sqrt{r^2 + C^2} \tag{60}$$

where C is a parameter.

Appropriate solutions of Poisson equation with given above radial basis function are:

Case 1

$$\phi(r) = \frac{r^{2+k}}{(2+k)^2} \tag{61}$$

Case 2

$$\phi(r) = \begin{cases} 0 \text{ for } r = 0 \\ \dfrac{r^{2k}(-2 + (2+k)\ln r)}{(2+k)^3} \text{ for } r \neq 0 \end{cases} \tag{62}$$

Case 3

$$\phi(r) = -\frac{C^3}{3} \ln\left(C\sqrt{r^2 + C^2} + C^2\right) + \frac{r^2 + 4C^2}{9}\sqrt{r^2 + C^2} \tag{63}$$

The fundamental solution of Poisson equation is function:

$$f_S(r) = \ln r \tag{64}$$

■ Numerical example

To prepare the numerical experiment three packages are proposed (Change the path to the actual location of the package in the import expression below to use the package). Some auxiliary function (to calculate coordinates of collocation points, the source points, to calculate the points for approximation by RBF) are defined. Procedures to find solution of Laplace equation, Poisson equation, Helmholtz equation with fundamental solution method and by using radial basis functions are included.

In[1]:=
```
<< "Anita‘trefftz‘"; << "Anita‘poisson‘"; << "Anita‘helmholtz‘";
```

The example solved by the combination of the methods described above is presented. The dimensionless parameters of considered region are: `a(=xmax)=1`, `b(=ymax)=1` and `c(xOpen)=0.2.` The uniform dimensionless pressure in porous medium $P_0 = 1$. The pressure outside the porous medium is $P_1 = 0.5$.

In[2]:=
```
p1 = 0.5; p0 = 1; xmin = 0.0; xmax = 1.0;
ymin = 0.0; ymax = 1.0; xOpen = 0.2;
```

```
General::spell1 :
 Possible spelling  error: new symbol name "ymin" is similar to
    existing symbol  \!\(xmin\) \". \!\(\*ButtonBox[\" More… \",
ButtonStyle -> \" RefGuideLinkText \", ButtonFrame -> None,
ButtonData :> \" General::spell1 \"]\) "
```

```
General::spell1 :
 Possible spelling  error: new symbol name "ymax" is similar to
    existing symbol  \!\(xmax\) \". \!\(\*ButtonBox[\" More… \",
ButtonStyle -> \" RefGuideLinkText \", ButtonFrame -> None,
ButtonData :> \" General::spell1 \"]\) "
```

```
General::spell1 :
 Possible spelling  error: new symbol name "xOpen" is
   similar to existing symbol  "Open". More…
```

```
General::stop : Further output of  General::spell1 will
   be suppressed during this  calculation. More…
```

The number of points (on unit boundary) for interpolation is:

In[3]:=
```
nm = 20;
```

Number of collocation points on unit boundary and number of collocation points on open edge are:

In[4]:=
```
nc = 20; ncOpen = xOpen * nc / (xmax - xmin);
```

Number of source points on unit boundary and distance between contour of the region and curve with source points are:

In[5]:=
```
ns = 20; s = 0.05;
```

The coordinates of points for approximation of right–hand side functions, source points are calculated with auxiliary functions:

```
In[6]:= {xm, ym, nma} = getXYMesh[xmin, xmax, ymin, ymax, nm];
        {xs, ys, nsa} = getXYSource[xmin, xmax, ymin, ymax, ns, s];
```

For first time step, $\Delta\tau$=1.3333, Helmholtz equation (23) with boundary conditions (24, 25) is solved using algorithm presented in previous section. The parameter k^2 of Helmholtz type equation is:

```
In[8]:= dt = 1.33333;
        k2 = Sqrt[1 / dt];
```

Right–hand side function is interpolated by RBF given by equation (46). For solution of equation (39) the function (51) is applied. The particular solution is a function named `up[x,y]`. For futher calculations partial differentials are requaired.

```
In[10]:= fr[x_, y_] := -p0 / dt;
         coefr = helmholtzParticular[xm, ym, nma, fr];
         up[x_, y_] := Sum[
            coefr[[k]] * psih[((x - xm[[k]]) ^2 + (y - ym[[k]]) ^2) ^0.5, k2],
            {k, 1, nma}];
         dxup[x_, y_] := Sum[coefr[[k]] *
             dxpsih[x, y, xm[[k]], ym[[k]], k2], {k, 1, nma}];
         dyup[x_, y_] := Sum[coefr[[k]] *
             dypsih[x, y, xm[[k]], ym[[k]], k2], {k, 1, nma}];
```

```
General::spell1 :
 Possible spelling  error: new symbol name "dyup" is similar to
    existing symbol  \!\(dxup\) \". \!\(\*ButtonBox[\" More… \",
ButtonStyle -> \" RefGuideLinkText \", ButtonFrame -> None,
ButtonData :> \" General::spell1 \"]\) "
```

The error of approximation is presented below:

In[15]:=
```
n = 9; dx = (xmax - xmin) / n; dy = (ymax - ymin) / n;
xy = Table[{xmin + (i - 1) * dx, ymin + (j - 1) * dy},
   {i, 1, n + 1}, {j, 1, n + 1}];
xyapprox = Table[fr[xy[[i, j, 1]], xy[[i, j, 2]]]
    - Sum[coefr[[k]]
         * rbfh[((xm[[k]] - xy[[i, j, 1]])^2 +
           (ym[[k]] - xy[[i, j, 2]])^2)^0.5],
     {k, 1, nma}], {i, 1, n + 1}, {j, 1, n + 1}];
f3 = ListPlot3D[xyapprox, AxesLabel → {"X", "Y", ""},
   Ticks → {{{n, "1"}}, {{ymax, "0"}, {n, "1"}}, Automatic},
   ViewPoint → {-2.5, -2, 1.5}];
```

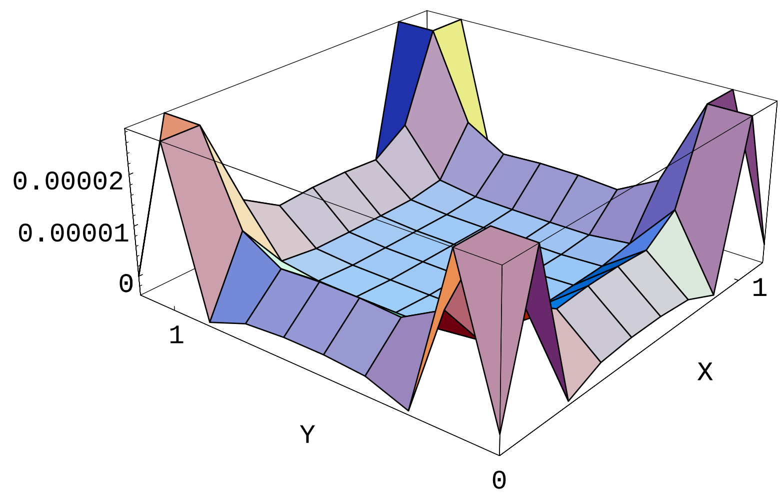


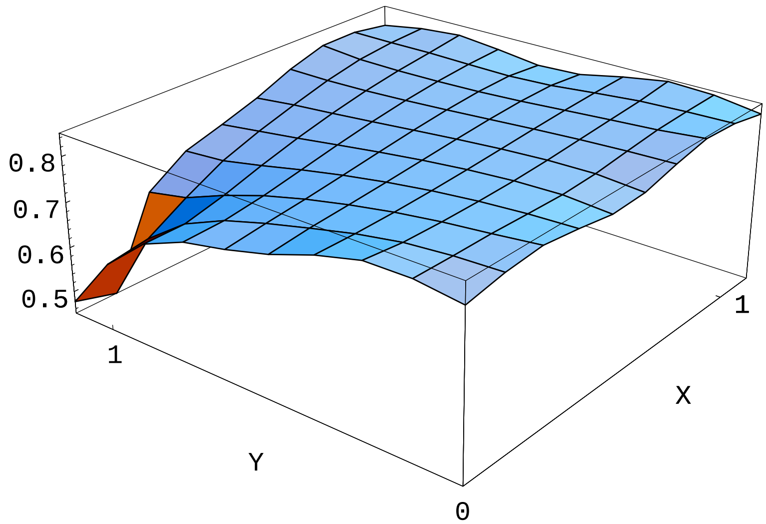
Figure 4 shows relative error of right–hand side fuction interpolated by RBF. Error of order 10^{-5} points a very good approximation of right–hand side function.

To find function described by formulas (44) and (45) the fundamental solution (56) is used. Linear combination of particular solution and solution based on fundamntal solutions is presented y function u[x,y]

In[19]:=
```
g[x_, y_] := p1;
coefun = helmholtzFundamental[xmin, xmax, ymin,
   ymax, nc, ncOpen, xs, ys, up, dxup, dyup, g, k2];
v[x_, y_] := Sum[coefun[[k]] * fsh[
     Sqrt[(xs[[k]] - x)^2 + (ys[[k]] - y)^2], k2], {k, 1, nsa}];
dxv[x_, y_] := Sum[coefun[[k]] *
    dxfsh[x, y, xs[[k]], ys[[k]], k2], {k, 1, nsa}];
dyv[x_, y_] := Sum[coefun[[k]] *
    dyfsh[x, y, xs[[k]], ys[[k]], k2], {k, 1, nsa}];
u[x_, y_] := up[x, y] + v[x, y];
dxu[x_, y_] := dxup[x, y] + dxv[x, y];
dyu[x_, y_] := dyup[x, y] + dyv[x, y];
```

Pressure distribution in the first time step is presented in Figure 5. The pressure is within the range (0.5,0.75), which confirms the flow gas out.

In[27]:=
```
xysolh = Table[u[xy[[i, j, 1]], xy[[i, j, 2]]],
    {j, 1, n + 1}, {i, 1, n + 1}];
plots = {ListPlot3D[xysolh, AxesLabel → {"X", "Y", ""},
     Ticks → {{{n, "1"}}, {{ymax, "0"}, {n, "1"}}, Automatic},
     ViewPoint → {-2.5, -2, 1.5}, PlotRange → {0.45, 0.85}]};
```

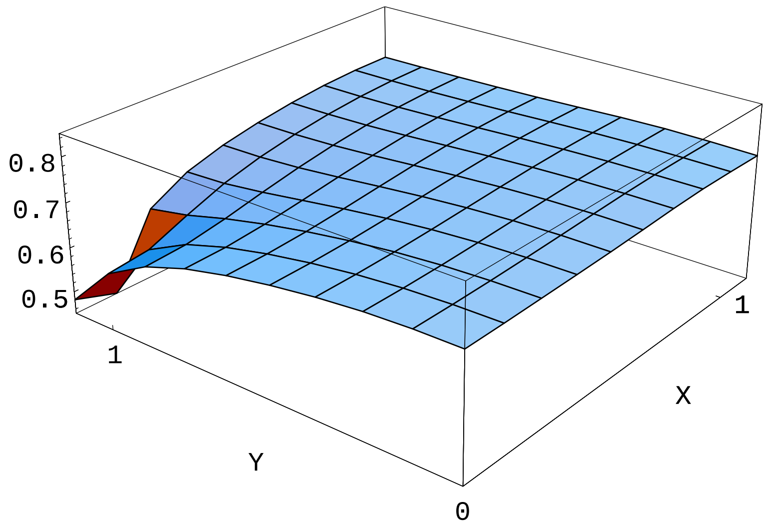


Results of the next time steps are included in Figure 6, which is animated graph.

In[29]:=
```
ntime = 1;
nmaxtime = 10;
eps = 10^-2;
While[ntime < nmaxtime,
  coefun0 = laplaceFundamental[xmin,
    xmax, ymin, ymax, nc, ncOpen, xs, ys, g, u];
  u0[x_, y_] := Sum[coefun0[[k]] *
     fsp[Sqrt[(x - xs[[k]])^2 + (y - ys[[k]])^2]], {k, 1, nsa}];
  dxu0[x_, y_] := Sum[coefun0[[k]] *
     dxfsp[x, y, xs[[k]], ys[[k]]], {k, 1, nsa}];
  dyu0[x_, y_] := Sum[coefun0[[k]] *
     dyfsp[x, y, xs[[k]], ys[[k]]], {k, 1, nsa}];
  xyapprox0 = Table[u0[xy[[i, j, 1]], xy[[i, j, 2]]],
    {j, 1, n + 1}, {i, 1, n + 1}];
  error = eps;
  iter = 1;
  While[(error ≥ eps ) && (iter ≤ 2),
   frp[x_, y_] := -1 / dt +
     (u0[x, y] / dt - dxu0[x, y]^2 - dyu0[x, y]^2) / u[x, y];
   coefr1 = poissonParticular[xm, ym, nma, frp];
   up1[x_, y_] := Sum[coefr1[[k]] * psip[
       ((x - xm[[k]])^2 + (y - ym[[k]])^2)^0.5], {k, 1, nma}];
   dxup1[x_, y_] := Sum[coefr1[[k]] *
      dxpsip[x, y, xm[[k]], ym[[k]]], {k, 1, nma}];
   dyup1[x_, y_] := Sum[coefr1[[k]] *
      dypsip[x, y, xm[[k]], ym[[k]]], {k, 1, nma}];
   coefun1 = poissonFundamental[xmin, xmax, ymin,
     ymax, nc, ncOpen, xs, ys, up1, dxup1, dyup1, g];
   xyapprox1 = Table[u1[xy[[i, j, 1]], xy[[i, j, 2]]],
     {j, 1, n + 1}, {i, 1, n + 1}];
   v1[x_, y_] := Sum[coefun1[[k]] * fsp[
       ((x - xs[[k]])^2 + (y - ys[[k]])^2)^0.5], {k, 1, nsa}];
   u1[x_, y_] := up1[x, y] + v1[x, y];
   error = Max[Abs[xyapprox0 - xyapprox1]];
   xyapprox0 = xyapprox1;
   coefr0 = coefr1;
   coefun0 = coefun1;
   u0[x_, y_] := Sum[
      coefr0[[k]] * psip[((x - xm[[k]])^2 + (y - ym[[k]])^2)^0.5],
      {k, 1, nma}] + Sum[coefun0[[k]] * fsp[
        ((x - xs[[k]])^2 + (y - ys[[k]])^2)^0.5], {k, 1, nsa}];
   dxu0[x_, y_] := Sum[coefr0[[k]] * dxpsip[x, y,
        xm[[k]], ym[[k]]], {k, 1, nma}] + Sum[
      coefun0[[k]] * dxfsp[x, y, xs[[k]], ys[[k]]], {k, 1, nsa}];
   dyu0[x_, y_] := Sum[coefr0[[k]] *
       dypsip[x, y, xm[[k]], ym[[k]]], {k, 1, nma}] + Sum[
      coefun0[[k]] * dyfsp[x, y, xs[[k]], ys[[k]]], {k, 1, nsa}];
   iter += 1;
  ];
  plots = Append[plots,
    ListPlot3D[xyapprox0, AxesLabel → {"X", "Y", ""},
     Ticks → {{{n, "1"}}, {{ymax, "0"}, {n, "1"}}, Automatic},
     ViewPoint → {-2.5, -2, 1.5}, PlotRange → {0.45, 0.85}]];
  coefr = coefr1;
  coefun = coefun1;
  u[x_, y_] :=
   Sum[coefr[[k]] * psip[((x - xm[[k]])^2 + (y - ym[[k]])^2)^0.5],
     {k, 1, nma}] + Sum[coefun[[k]] * fsp[
       ((x - xs[[k]])^2 + (y - ys[[k]])^2)^0.5], {k, 1, nsa}];
  dxu[x_, y_] := Sum[coefr[[k]] * dxpsip[x, yxm[[k]], ym[[k]]],
     {k, 1, nma}] + Sum[
     coefun[[k]] * dxfsp[x, y, xs[[k]], ys[[k]]], {k, 1, nsa}];
  dyu[x_, y_] := Sum[coefr[[k]] * dypsip[x, y, xm[[k]], ym[[k]]],
     {k, 1, nma}] + Sum[
     coefun[[k]] * dyfsp[x, y, xs[[k]], ys[[k]]], {k, 1, nsa}];
  ntime += 1;
 ];
```

```
General::spell1 :
 Possible spelling  error: new symbol name "dyu0" is similar to
    existing symbol \!\(dxu0\) \". \!\(\*ButtonBox[\"More…\",
ButtonStyle->\"RefGuideLinkText\", ButtonFrame->None,
ButtonData:>\"General::spell1\"]\) "
```

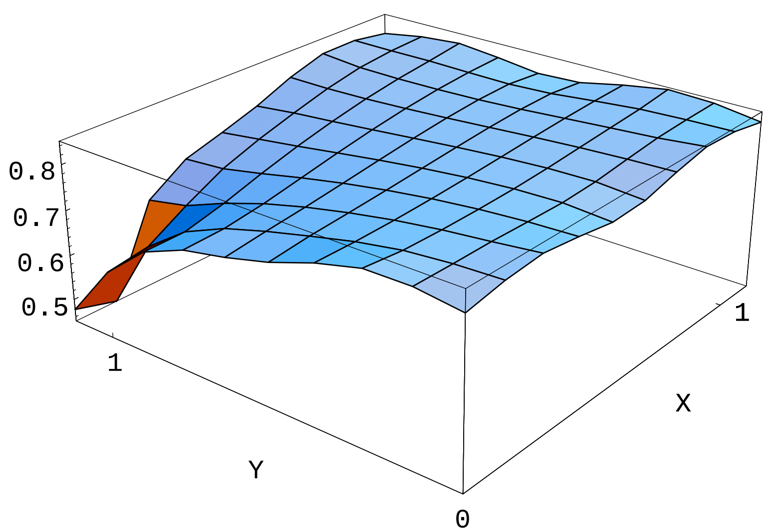
```
General::spell1 :
 Possible spelling  error: new symbol name "dyup1" is
   similar to existing symbol  "dxup1". More…
```



The animation shows flow out of gas from the considered region. Values of the pressure inside the considered region are very close to value of the pressure outside the reservoir.

In[33]:= **<< Graphics'Animation'**

In[34]:= **ShowAnimation[plots]**



■ Conclusions

In our paper the Trefftz method is used for solving time dependent phenomena. The finite difference method is implemented to approximate differential with respect to time parameter. The technique for obtaining solution of inhomogeneoust Helmholtz equation is applied. The inhomogeneous nonlinear Poisson equation given in implicit form is solved using the iterative technique. To find solution of the one iteration step the Trefftz method for inhomogenoues Poisson equation has been implemented. Obtained result shows gas outflow from porous medium. The numerical experiment confirms that the chosen meshless method is useful for solving dynamical problems.

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