Teaching Mathematics and Doing history of mathematics

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I am teacher of Mathematics and searcher in history of mathematics. I use *Mathematica* for teaching some elements of Mathematics (See [Verdier)] but I use *Mathematica* too for my researches in history of Mathematics. Each example, I will give in this talk, is a way of explorating theses fields : Mathematics, History of Mathematics, Teaching of Mathematics and (of course!) *Mathematica*. A way of asking some questions (my questions or student's questions) about an object of teaching and researching : *Mathematica.* A way of giving some answers.

In history, I study a journal of mathematics : the "Journal des Mathématiques pures et appliquées" (See http://math−doc.ujf−grenoble.fr/JMPA/). This journal was founded in 1836 by Joseph Liouville (1809−1882). Liouville was responsible for this publication during 39 years, between 1836 and 1874. So, the name of this journal is "Liouville's journal". Each example is extracted from Liouville's journal.

■ Integrating with Liouville and with *Mathematica*

This example will be extracted from Liouville's work and will be gived to my students as this form : How to calculate actually these integrals without and with *Mathematica* ?

a) $I = \int_0^\infty \frac{\ln(x)}{1+x^2}\, dx$? b)$J = \int_0^\infty \frac{\ln(x)}{a^2+x^2}\, dx$? c) $K = \int_0^\infty \int_0^\infty \frac{\ln(xy)}{(a^2+x^2)(b^2+y^2)}\, dx\, dy$ d) Give a generalization ?

If you try with *Mathematica* you will obtain for I :

```
Integrate[Log[x] / (1 + x^2), {x, 0, Infinity}]
```

```
0
```

For having this result (after beeing sure that I is a definite integral), the best solution is to do a partition of the integral : $\int_0^\infty \frac{\ln(x)}{1+x^2}\, dx = \int_0^1 \frac{\ln(x)}{1+x^2}\, dx + \int_1^\infty \frac{\ln(x)}{1+x^2}\, dx$. Let I1 the first end I2 the second. In the second integral, if you do y=1/*x*, you will have : I2 = – I1 and I = 0.

For J :

```
Integrate[Log[x] / (a^2 + x^2), {x, 0, Infinity}]
```

$$-\frac{I\left(\mathrm{Log}\left[-\frac{I}{a}\right]^2 - \mathrm{Log}\left[\frac{I}{a}\right]^2\right)}{4\,a}$$

If you give a numerical value to *a*, you will have :

```
Integrate[Log[x] / (3^2 + x^2), {x, 0, Infinity}]
```

$\frac{1}{6}\pi$ Log[3]

Why all these curious answers ? If you know complex analysis, it is easy to understand! Here we can say that a is a positive number so "Log(I/a) =–Log(a) + j Pi/2 and Log(–i/a) = –Loga – j Pi/2". After some simplifications, you will find : J = $\frac{\pi \ln(a)}{2a}$. When you read Liouville's notebook (See [Liouville, MS 36 20(4)]), you can see that Liouville is agree with the first calculus. For the second, he uses no calculus. He uses a decomposition : $\int_0^\infty \frac{\ln(x)}{a^2+x^2}\,dx = \int_0^1 \frac{\ln(x)}{a^2+x^2}\,dx + \int_1^\infty \frac{\ln(x)}{a^2+x^2}\,dx$. After, he does : $y = 1/\mathrm{x}$ in the second integral and he obtains this result : $\int_0^\infty \frac{\ln(x)}{a^2+x^2}\,dx = \frac{\pi \ln(a)}{2a}$. He proposes a generalization :

$$\int_0^\infty \int_0^\infty \frac{\ln(x\,y)}{(a^2+x^2)\,(b^2+y^2)} \ldots dx\,\mathrm{dy} \ldots = \frac{(\pi/2)^n \ \ln(\mathrm{ab}...)}{\mathrm{ab..}}.$$

We can try with numerical values (for example $a = \pi$; $b = 2$ and $n = 2$) :

```
Integrate[(Log[x y]) / ((Pi^2 + x^2) (2^2 + y^2)),
 {x, 0, Infinity}, {y, 0, Infinity}]
```

$\frac{1}{8}\pi$ Log[2 π]

Mathematica is agree with Liouville for these values. Are you agree with Liouville ? This example will be choosed in order to see that *Mathematica* is, of course, an object of calculus, but, for understanding *Mathematica*, you have to know Mathematics. Often, it deals with not elementary mathematics. Here complex analysis (See [Dieudonné, 1980], pp. 255, ... for example). For finishing this example, a question of one of my students : "Here the problem is that, for *Mathematica* , a is a general number (eventually) a complex. Is it possible to tell to *Mathematica* that a is a real number in order to have a correct calculus without using complex analysis ?"

■ Duhamel's series

In a Duhamel's article(Cf. [Duhamel, 1839]), we have a proof ot the convergence of $\sum_{n=2}^{\infty} \left(\frac{\ln n}{n}\right)^2$ (thanks to Raabe's and Duhamel's criterium). It is possible to have an exact calculus ? A generalization ?

With *Mathematica*, we find :

```
Sum[(Log[n] / n)^2, {n, 2, Infinity}]
```

Zeta″[2]

In fact, it is not difficult to give an idea about the demonstration : by derivation of $\zeta(x)$, you obtain $\zeta''(x)$. If you choose $x = 2$, you will find Duhamel's serie. In order to have a generalization, you can calculate some series :

```
Table[Sum[(Log[n] / n)^p, {n, 2, Infinity}], {p, 2, 10}]
```

{Zeta″[2], –Zeta$^{(3)}$[3], Zeta$^{(4)}$[4], –Zeta$^{(5)}$[5], Zeta$^{(6)}$[6], –Zeta$^{(7)}$[7], Zeta$^{(8)}$[8], –Zeta$^{(9)}$[9], Zeta$^{(10)}$[10]}

It seems to be correct to propose : $\sum_{n=2}^{\infty} \left(\frac{\ln n}{n}\right)^p = (-1)^p\, \zeta^{(p)}(\mathrm{p})$

Of course, it is not a demonstration. Just a proposition. The demonstration is not very difficult, by derivation of ζ. If you do not know Riemann's function ζ (if you know it is the same case), you will want to have an approximation of this calculus. Well, you will try :

```
Table[NSum[(Log[n] / n)^p, {n, 2, Infinity}], {p, 2, 10}]
```

```
{1.98928, 0.374044, 0.0990074, 0.0295015, 0.00931102,
 0.00303758, 0.00101218, 0.00034223, 0.000116945}
```

If you try by an other way (for $p = 10$, for example) :

```
Sum[(Log[n] / n)^10, {n, 2, Infinity}] // N
```

```
-22898.6
```

It is different from

```
NSum[(Log[n] / n)^10, {n, 2, Infinity}]
```

```
0.000116945
```

What happened with ζ ?

```
D[Zeta[x], {x, 10}] /. x -> 10 // N
```

```
-22898.6
```

What is correct ? What happened ? Clearly, the last result is completly wrong! It is due to ζ, which is not a "common function".

■ Davis'primes

Davis is a Liouville's author. Just a remark : I have no information about this mathematician. A mathematician from England ? I don't know and *Mathematica* will don't give any information about that! Davis wrote only one article in Liouville's journal. It is not exactly an article : it is a letter to Liouville (See [Davis, 1866]). He gave a list of prime numbers. In his notebook, Liouville wrote : " Il s'agit comme on voit de nombres premiers dont la valeur surpasse cent millions. M$^{\mathrm{r}}$ William Davis nous écrit qu'il a cherché tous ceux qui existent de 100 00 00 0 1 à 1 0000 1699 et qu'il en a trouvé quatre–vingt dix neuf. Nous allons donné les quatre derniers chiffres des nombres premiers ainsi construits de sorte qu'il faudra placer l'unité suivie de quatre zéros (ou 1 0000) en tête de ces quatre derniers chiffres pour avoir les nombres eux–mêmes. 0007, 0013, 0037, etc" (See [Liouville, MS 36 32 (4)]). It is easy to know if Davis is right.

```
PrimeQ[100000007]
```

```
True
```

But for the second number, we find :

```
PrimeQ[100000013]
```

```
False
```

It is not a prime number although Davis thought that. Liouville was a very scrupulous reader of "his" authors. Not in this case. We find an error in Davis'article. Is is other errors ? He thought that between 100 00 00 0 1 and 1 0000 1699, there is 99 prime numbers. Right ?

```
Count[Table[PrimeQ[i], {i, 100000001, 100001699}], True]
```

```
91
```

False Mr Davis! There is only 91 prime numbers! Where are Davis'errors ? In order to extract prime numbers between 100 00 00 0 1 and 1 0000 1699, I create a criterium :

```
Clear[crit]; crit[x_] := PrimeQ[x]
```

Now, I am able to extract Davis'primes :

```
Clear[davisprimes];
davisprimes = Select[Table[i, {i, 100000001, 100001699}], crit]
```

```
{100000007, 100000037, 100000039, 100000049,
 100000073, 100000081, 100000123, 100000127,
 100000193, 100000213, 100000217, 100000223,
 100000231, 100000237, 100000259, 100000267,
 100000279, 100000357, 100000379, 100000393, 100000399,
 100000421, 100000429, 100000463, 100000469, 100000471,
 100000493, 100000541, 100000543, 100000561, 100000567,
 100000577, 100000609, 100000627, 100000643, 100000651,
 100000661, 100000669, 100000673, 100000687, 100000717,
 100000721, 100000793, 100000799, 100000801, 100000837,
 100000841, 100000853, 100000891, 100000921, 100000937,
 100000939, 100000963, 100000969, 100001029, 100001053,
 100001059, 100001081, 100001087, 100001107, 100001119,
 100001131, 100001147, 100001159, 100001177, 100001183,
 100001203, 100001207, 100001219, 100001227, 100001303,
 100001329, 100001333, 100001347, 100001357, 100001399,
 100001431, 100001449, 100001467, 100001507, 100001533,
 100001537, 100001569, 100001581, 100001591, 100001611,
 100001623, 100001651, 100001653, 100001687, 100001689}
```

Thanks to a comparison with Davis'letter, my little girl Imène (6 years old) founded the eight errors : 100000013, 100000391,100000657,100000723,100001221,100001353, 100001549, 100001647 are not prime numbers! For having a confirmation :

```
Divisors [{100000013, 100000391, 100000657,
  100000723, 100001221, 100001353, 100001549, 100001647}]
```

```
{{1, 827, 120919, 100000013}, {1, 2153, 46447, 100000391},
 {1, 2711, 36887, 100000657}, {1, 1447, 69109, 100000723},
 {1, 2689, 37189, 100001221}, {1, 7247, 13799, 100001353},
 {1, 2447, 40867, 100001549}, {1, 5737, 17431, 100001647}}
```

Imène is right!
Two remarks about this work :
– Liouville does not see Davis'errors. I think, he was not realy interested in Davis'results but only in Davis'methods for having theses results : In his, notebook, he wrote : "Il paraît n'avoir suivi à cet effet la méthode (connue) qui consiste à essayer la division par les premiers inférieurs à la racine carrée des nombres c'est à dire ici inférieurs à 10 000." (See [Liouville, MS 36 32 (4)]. Eleven years later, in one of his notebook, Liouville wrote : "M.William Davis (J.de mathématiques T.?P? la deuxième série) donne des nombres premiers ... mais ces nombres sont surpassés par celui d'Euler (Journal P? du tome XI, 2 ème série, 1866)" (See [Liouville, MS 36 37 (10)]]). Liouville gave a question to historians : Euler gave a list of (good) prime numbers more interesting than Davis'list ?
– In order to have davis'list, it would be a common idea to do that :
for having the first prime numbers you can do :

```
Prime /@ Range[100]
```

```
{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53,
 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109,
 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179,
 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241,
 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313,
 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389,
 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461,
 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541}
```

For having the prime numbers between Prime[50] and Prime[100], you can do :

```
Take[Prime /@ Range[100],
 - Length[Prime /@ Range[100]] + Length[Prime /@ Range[50]]]
```

```
{233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307,
 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383,
 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457,
 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541}
```

For our problem, the fist Davis'prime is 100000007, the last is 100001689. And :

```
{Prime[100000007], Prime[100001689]}
```

```
{2038074853, 2038111111}
```

It would be a solution to try :

```
Take[Prime /@ Range[Prime[100001689]],
 - Length[Prime /@ Range[Prime[100001689]]] +
  Length[Prime /@ Range[Prime[100000007]]]]
```

```
Out of memory.  Exiting.
```

A solution. But a very bad solution! "Out of memory. Exiting"!

■ Liouville's authors

Before giving examples of "Doing history of mathematics thanks to *Mathematica*", I will give some generalities. For doing history (of mathematics), you have to create corpus and thanks to analysis of your corpus you can extract some elements and having an historian's talk. Often, corpus have a form of a list of objects (authors'names, number of articles, etc). And *Mathematica* has a lot of possibilities in order to do operations with lists : extracting, sorting, etc. In this opinion, *Mathematica* is the hand of the historian! I will give here just some elementary examples, about Liouville's authors. I will present here a simplify model.

200 authors wrote in Liouville's journal between 1836 and 1874. For each author I know number of articles and number of pages. So I have a list of authors with their publications. How to study the dynamic of this journal ? I do like Liouville (every period of 5 years is a Liouville's period). The first serie is between 1836 and 1855. Liouville decided to create an other serie in 1856. So the first period A is [1836–1840], the second period B is [1841–1845], ..., the last period (H) is [1871–1874].
I create a list of authors. Here is a "very" small database (an extract) :

```
Clear[Liste];
Liste =  {{Liouville, 41, 26, 19, 24, 74, 161, 21, 3} ,
  {Raabe, 0, 1, 0, 0, 0, 0, 0, 0}, {Duhamel, 3, 1, 2,
   3, 1, 0, 0, 0}, {Davis, 0, 0, 0, 0, 0, 0, 1, 0},
  {Gascheau, 0, 2, 0, 0, 0, 0, 0, 0}, {Puiseux, 0, 5, 5,
   3, 2, 2, 1, 1}, {Blanchet, 1, 4, 0, 0, 0, 0, 0, 0},
  {Cauchy, 5, 1, 1, 0, 0, 0, 0, 0}, {Hirst, 0, 0, 0, 0,
   2, 0, 0, 0 }, {Thomson, 0, 4, 2, 1, 0, 0, 0, 0}}
```

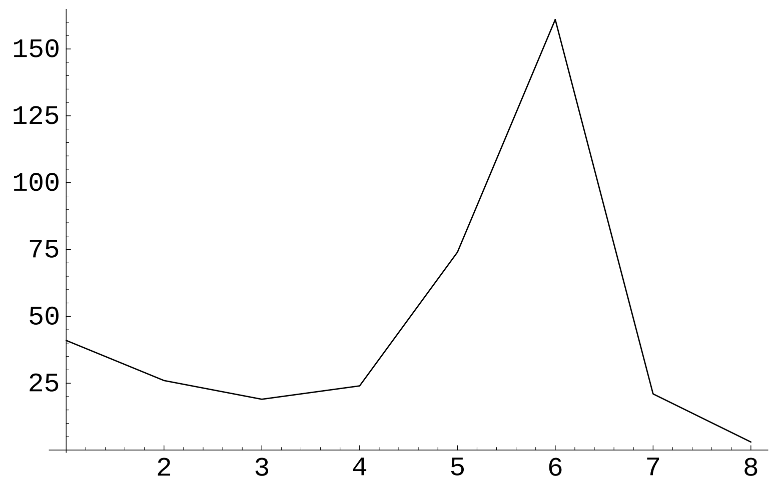
```
{{Liouville, 41, 26, 19, 24, 74, 161, 21, 3},
 {Raabe, 0, 1, 0, 0, 0, 0, 0, 0}, {Duhamel, 3, 1, 2, 3, 1, 0, 0, 0},
 {Davis, 0, 0, 0, 0, 0, 0, 1, 0}, {Gascheau, 0, 2, 0, 0, 0, 0, 0, 0},
 {Puiseux, 0, 5, 5, 3, 2, 2, 1, 1},
 {Blanchet, 1, 4, 0, 0, 0, 0, 0, 0},
 {Cauchy, 5, 1, 1, 0, 0, 0, 0, 0}, {Hirst, 0, 0, 0, 0, 2, 0, 0, 0},
 {Thomson, 0, 4, 2, 1, 0, 0, 0, 0}}
```

Question 1 : How to have for each author his dynamic of publications in Liouville's journal ? For each author, I create a ListPlot in order to "see" his dynamic of publication.

```
Clear[dynpubli]; dynpubli[x_] :=
 ListPlot[Liste[[Flatten[Position[Liste, x]][[1]]]] // Rest
, PlotJoined -> True]
```

Example :

```
dynpubli[Liouville]
```

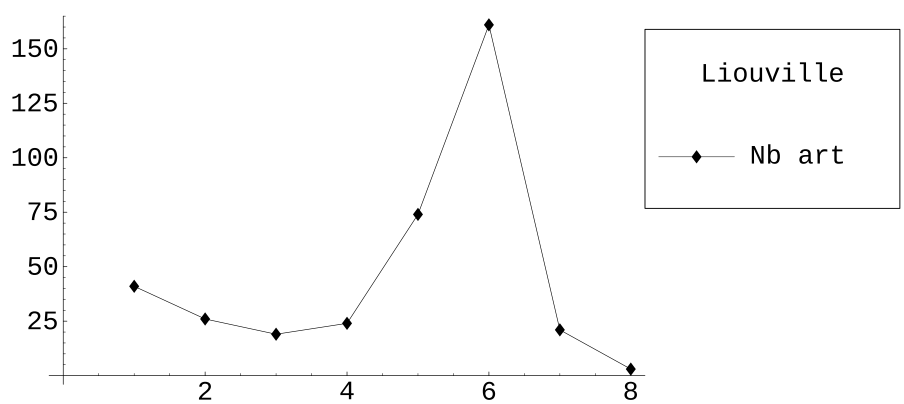



```
- Graphics -
```

Of course, it is possible to improve the presentation! With the package Graphics‘MultipleListPlot. Just one example of sophistication :

```
<< Graphics‘MultipleListPlot‘

MultipleListPlot[
 Liste[[Flatten[Position[Liste, Liouville]][[1]]]] // Rest,
 LegendLabel -> "Liouville", PlotLegend -> {"Nb art"
                 }, PlotJoined -> True,
 LegendPosition -> {1., 0.01}]
```




```
- Graphics -
```

Question 2 : How to select all the authors of one period ? (problem of extraction). In this database we have 8 periods; so I create a criterium for extracting authors of the first period.

For extracting, the authors of the fist period (authors who write one article (ore more) during the period A (1836–1840), I choose authors who did not write :

```
Cases[Liste, {A_, 0, b___}]
```

```
{{Raabe, 0, 1, 0, 0, 0, 0, 0, 0},
 {Davis, 0, 0, 0, 0, 0, 0, 1, 0}, {Gascheau, 0, 2, 0, 0, 0, 0, 0, 0},
 {Puiseux, 0, 5, 5, 3, 2, 2, 1, 1},
 {Hirst, 0, 0, 0, 0, 2, 0, 0, 0}, {Thomson, 0, 4, 2, 1, 0, 0, 0, 0}}
```

And after, thanks to the complement, we have a the list of A– authors (authors of the period A) :

```
Complement[Liste, %]
```

```
{{Blanchet, 1, 4, 0, 0, 0, 0, 0, 0},
 {Cauchy, 5, 1, 1, 0, 0, 0, 0, 0}, {Duhamel, 3, 1, 2, 3, 1, 0, 0, 0},
 {Liouville, 41, 26, 19, 24, 74, 161, 21, 3}}
```

Of course, for the other periods, it the same thing! For the C–Authors, we have :

```
Complement[Liste, Cases [Liste, {A_, a_, b_, 0, c___}]]
```

```
{{Cauchy, 5, 1, 1, 0, 0, 0, 0, 0}, {Duhamel, 3, 1, 2, 3, 1, 0, 0, 0},
 {Liouville, 41, 26, 19, 24, 74, 161, 21, 3},
 {Puiseux, 0, 5, 5, 3, 2, 2, 1, 1},
 {Thomson, 0, 4, 2, 1, 0, 0, 0, 0}}
```

Question 3 (Last question) : How to do for having a command in order to sort, by number of publication, the list of A–Authors, or B–authors, etc ?

For sorting the elements of list into canonical order, it is easy with Sort :

```
Sort[Liste]
```

```
{{Blanchet, 1, 4, 0, 0, 0, 0, 0, 0},
 {Cauchy, 5, 1, 1, 0, 0, 0, 0, 0}, {Davis, 0, 0, 0, 0, 0, 0, 1, 0},
 {Duhamel, 3, 1, 2, 3, 1, 0, 0, 0},
 {Gascheau, 0, 2, 0, 0, 0, 0, 0, 0}, {Hirst, 0, 0, 0, 0, 2, 0, 0, 0},
 {Liouville, 41, 26, 19, 24, 74, 161, 21, 3},
 {Puiseux, 0, 5, 5, 3, 2, 2, 1, 1},
 {Raabe, 0, 1, 0, 0, 0, 0, 0, 0}, {Thomson, 0, 4, 2, 1, 0, 0, 0, 0}}
```

For sorting by number of publications, we can use an algorithm of Verdel's book (See [Verdel],pp. 80). I will adapt this program, in order to sort a list of lists. For example, for sorting the C–Authors :

```
% //. {x___, y_, z_, v___} /; y[[4]] > z[[4]] -> {x, z, y, v}
```

```
{{Blanchet, 1, 4, 0, 0, 0, 0, 0, 0},
 {Davis, 0, 0, 0, 0, 0, 0, 1, 0}, {Gascheau, 0, 2, 0, 0, 0, 0, 0, 0},
 {Hirst, 0, 0, 0, 0, 2, 0, 0, 0}, {Raabe, 0, 1, 0, 0, 0, 0, 0, 0},
 {Cauchy, 5, 1, 1, 0, 0, 0, 0, 0}, {Duhamel, 3, 1, 2, 3, 1, 0, 0, 0},
 {Thomson, 0, 4, 2, 1, 0, 0, 0, 0},
 {Puiseux, 0, 5, 5, 3, 2, 2, 1, 1},
 {Liouville, 41, 26, 19, 24, 74, 161, 21, 3}}
```

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