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2D Non-central Forces and Undiscovered Orbitals

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Abstract

In a polar coordinate system we consider fifteen classes of forces resulting in countless undiscovered orbitals. The classic Keplerian forces and its accompanied conic section orbitals are one of the special subclasses. Aside from the common theoretical foundation, the characteristics of the individual orbitals are given by the solution of corresponding equations of motion. These are nonlinear coupled differential equations. Solving these equations numerically and utilizing a Computer Algebra System such as Mathematica is conducive to the trajectories. Simulation of the trajectories provides a visual understanding about the motion under the influence of the generalized non-central forces. Our current investigation is a continuation of our keen interest in nonlinear physics phenomena. This report would be stimulating to individuals interested in physics in general and to those involved in classical mechanics in particular. In this report we include ample examples and their accompanied Mathematica codes. Utilizing these codes an interested reader may produce our findings and launch his/her own investigation.

Motivation and Goals

In our previous work we investigated the motion of a massive point-like particle under the influence of semi generalized central forces [1,2]. In a polar coordinate system we consider forces that are merely radial and distance dependent. The scope of the investigation is $\vec{F}(r) \sim r^n \hat{r}$ where n is within $-4 \le n \le 2$. This specific range includes two particular instances, namely $n = -2$ and 1. The former specifies the gravity and electrostatic forces i.e. the Keplerian forces, and the latter is merely a linear force. Consequently, the former gives the classic conic orbitals and the latter induces the *Elliptic* Harmonic orbits. The rest of the suggested n values gives peculiar orbits discussed in [2]. In our approach we outline the general theoretical foundation; as such one has the option of selecting unrestricted values for n including reals. Motivated with the outcome of our study we craft our current analysis. This augments our previous work in three major frontiers. First, we consider radial forces that are not merely distance dependent. Forces such as, $\vec{F} = f(\theta)\,\hat{r}$ and $\vec{F} = f(r, \theta)\,\hat{r}$, here θ is the polar angle. The f's are *arbitrary* functions, consequently there are *unlimited* corresponding orbitals. Second, in a polar coordinate system we consider forces that are merely azimuthal such as, $\vec{F} = g(r)\,\hat{\theta}$, $\vec{F} = g(\theta)\,\hat{\theta}$ and $\vec{F} = g(r, \theta)\,\hat{\theta}$. Similar to the previous case $g's$ are arbitrary functions, therefore there are *unlimited* orbitals. Third, we envision two-component forces, i.e. forces with radial and azimuthal components. A complete set of such forces are tabulated in Table 1. The table includes fifteen classes. These are called classes because each one embodies countless forces; these are labeled $f_{\rm ij}$ and $g_{\rm ij}$. The main objective of our investigation is to apply the classical mechanics analyzing the orbitals of a massive point-like particle undergoing the influence of tabulated forces in Table 1. This work is composed of four sections. In addition to Motivation and Goals, in Section 2 we outline the general foundation of our analysis. In Section 3 for a few random cases we simulate the orbitals. We close the work with concluding remarks.

Analysis of the Physics Problem

Following the objectives outlined in the previous section we consider the kinematics of a mobile massive point-like object of mass m in a two-dimensional space. Utilizing the polar coordinate system the acceleration is [3,4],

$$\ddot{\vec{r}} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{r} + \left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)\hat{\theta} \tag{1}$$

where according to the standard convention a single-dot and a double-dot are the first and the second derivatives with respect to time, respectively. Applying (1) the equation of motion is,

$$m\,\ddot{\vec{r}} = \vec{F} \tag{2}$$

The RHS of (2) is one of the fifteen cells of Table 1. Accordingly, (2) is a representative equation of motion. For instance in our previous work [1,2] we consider the impact of the content of cell$_{12}$; namely $\vec{F} = f_{12}(r)\,\hat{r}$. More specifically, we consider $f_{12}(r) \sim r^n$. For this class of forces the absence of the azimuthal component of the force assists in decoupling the associated equations resulting a single differential equation. Solution of the latter gives the desired orbitals. In general applying the latter procedure yields (2),

$$\begin{cases} m\left(\ddot{r} - r\,\dot{\theta}^2\right)_{\rm ij} = f_{\rm ij} & \text{for } i, j = 1, \ldots 4 \\ m\left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)_{k\ell} = g_{k\ell} & \text{for } k, \ell = 1, \ldots 4 \end{cases} \tag{3}$$

The set of equations given in (3) are coupled ODEs. For arbitrary functions such as f_{ij} and $g_{k\ell}$ most likely (3) is nonlinear. Unless otherwise for appealing cases one doesn't seek for their analytic solutions. Utilizing *Mathematica* [5] we solve (3) numerically.

Table 1. Symbolic representation of fifteen possible classes of forces with components along the radial and azimuthal directions in a polar coordinate system.

$\vec{F}$	1	2	3	4
1	$0_{11}\,\hat{r} + 0_{11}\,\hat{\theta}$	$f_{12}\,(r)\,\hat{r} + 0_{12}\,\hat{\theta}$	$f_{13}\,(\theta)\,\hat{r} + 0_{13}\,\hat{\theta}$	$f_{14}\,(r,\,\theta)\,\hat{r} + 0_{14}\,\hat{\theta}$
2	$0_{21}\,\hat{r} + g_{21}(r)\,\hat{\theta}$	$f_{22}\,(r)\,\hat{r} + g_{22}\,(r)\,\hat{\theta}$	$f_{23}\,(\theta)\,\hat{r} + g_{23}\,(r)\,\hat{\theta}$	$f_{24}\,(r,\,\theta)\,\hat{r} + g_{24}\,(r)\,\hat{\theta}$
3	$0_{31}\,\hat{r} + g_{31}(\theta)\,\hat{\theta}$	$f_{32}\,(r)\,\hat{r} + g_{32}\,(\theta)\,\hat{\theta}$	$f_{33}\,(\theta)\,\hat{r} + g_{33}\,(\theta)\,\hat{\theta}$	$f_{34}\,(r,\,\theta)\,\hat{r} + g_{34}(\theta)\,\hat{\theta}$
4	$0_{41}\,\hat{r} + g_{41}(r,\,\theta)\,\hat{\theta}$	$f_{42}\,(r)\,\hat{r} + g_{42}\,(r,\,\theta)\,\hat{\theta}$	$f_{43}(\theta)\,\hat{r} + g_{43}\,(r,\,\theta)\,\hat{\theta}$	$f_{44}\,(r,\,\theta)\,\hat{r} + g_{44}\,(r,\,\theta)\,\hat{\theta}$

Results

In this section according to Table 1 for a hand full of cases we investigate the specifics of the motion. Generally speaking, (3) is a set of coupled differential equations, aside from the force constant (see next paragraph) their numeric solutions require a set of four initial conditions. As one expects, the solutions are sensitive to the force strength and the initial conditions. Solutions are expressed as $\{r(t),\,\theta(t)\}$. Utilizing these we identify the orbitals and quantities of interest. We give examples highlighting the specifics.

Example 1. Consider the well known gravity and electrostatic force i.e. the Keplerian forces. The force falls in the category of cell_{12} of Table 1. We substitute $\frac{1}{m}f_{12}$with $-\xi\,\frac{1}{r^2}$; here ξ is the force constant. For instance, in the case of gravity the value of ξ is independent of m and is $G\,M$, where G is the universal gravity constant and M is the central mass. In the case of charge-charge interaction ξ is $K\,\frac{Q\,q}{m}$ where K is the electric coupling constant and Q, q are the charges of the point-like charges. By trial and error the initial conditions and the force constant are adjusted so that the orbital is a perfect stable circle. One such set of parameters is given in the figure caption of Fig 1a. It is reassuring that the output of our current approach is the same as [2]. It is noteworthy mentioning that one of the objectives of our current investigation is to numerically solve more challenging issues, see examples 2,3 and 4; where in [2] compatible with its objectives the emphasis was on less generalized case studies.

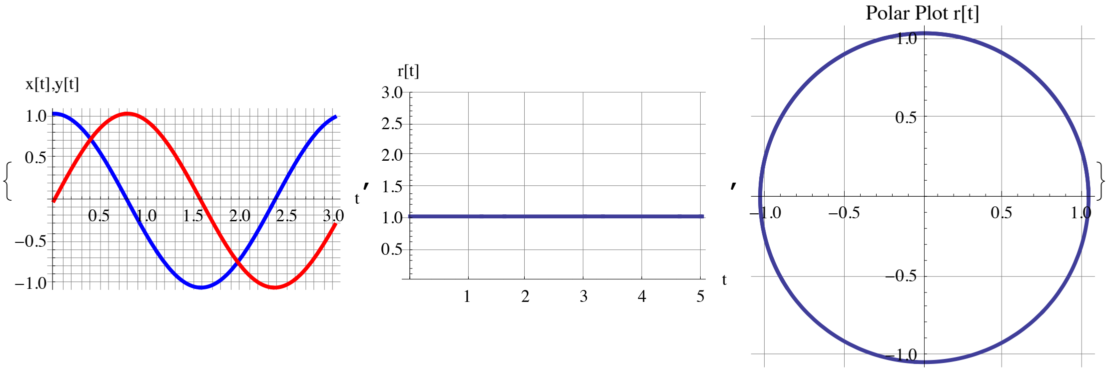

Figure 1a. The orbital is a perfect stable circle. The force strength is $\xi = 4.5$, and the initial values are $\{r(0),\,\dot{r}(0),\,\theta(0),\,\dot{\theta}(0)\} = \{1.04,\,0,\,0,\,2.2\}$

Figure 1a is comprised of three panels. The left plot displays the Cartesian components of the particle position. These are $\{x(t),\,y(t)\} = \{r(t)\cos[\theta(t)],\,r(t)\sin[\theta(t)]\}$. The plot shows shifted but identical oscillations of the individual components; the red curve is the x(t) and the blue curve is the y(t), respectively. The middle graph is the display of the radial distance of the particle as a function of time, t. It shows the distance of the particle from the origin is constant. Consequently its polar plot shown in the right panel exhibits a stable circular orbit. This is a classic and well-known result. Its confirmation within the body of our current work creates the forum for the rest of the investigation.

By adjusting the initial radial velocity the stable circular orbit of Fig 1a becomes a perfect, stable elliptical orbit. Figure caption of Fig 1b includes the specifics of the force strength and the initial conditions. The left plot of Fig 1b seemingly is similar to the corresponding Fig 1a. However, there are minor, hard to see differences. These are shown in the middle plot of Fig 1b. Its mildly wobbling behavior warrants non-circular, elliptic orbit shown in the right panel of Fig 1b.

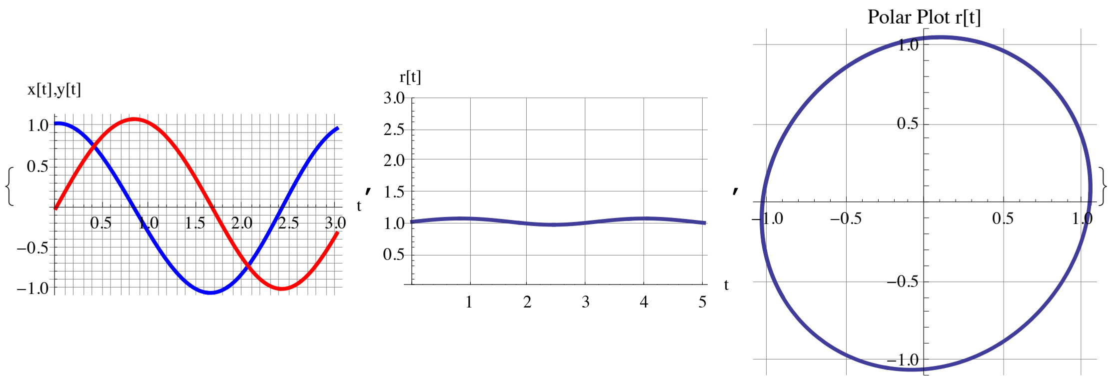

Figure 1b. A perfect stable ellipse. The force strength is $\xi = 4.5$, and initial values are $\{r(0), \dot{r}(0), \theta(0), \dot{\theta}(0)\} = \{1.04, 0.1, 0, 2.0\}$

In this case too our current output coincides with our previous work [2].

Example 2. Here we consider an example associated with the cell_{13}. We equate $\frac{1}{m} f_{13} \equiv -\xi\, \text{Sin}[\theta(t)]$; where ξ is the force constant. Aside from given aforementioned comments concerning ξ the value of m along with the other force related quantities, similar to the ones in the Keplerian case is being absorbed in ξ. This example is similar to the previous examples, its pure radial oriented character conserves the angular momentum of the particle. It is noteworthy mentioning that according to one of our objectives, the intent is to augment the body of knowledge concerning the motion under the influence of noncentral forces. Theoretically speaking any reasonable functional form for such forces should work. One such force e.g. Sin[θ] is the subject of example 2. There is nothing special about this suggested function, as such any function sustaining azimuthal characteristics would work as well. Specifically speaking, our suggested approach concerning augmenting the scope of the motion under the noncentral forces is a fresh theoretical idea. As such, we do not necessarily need to seek for a real-life evidence for its existence. As a result our suggested proposal and its solution paves the road for the "what-if scenarios."
For a set of parameters specified in the figure caption of Fig 2 we display graphic information including the orbital.

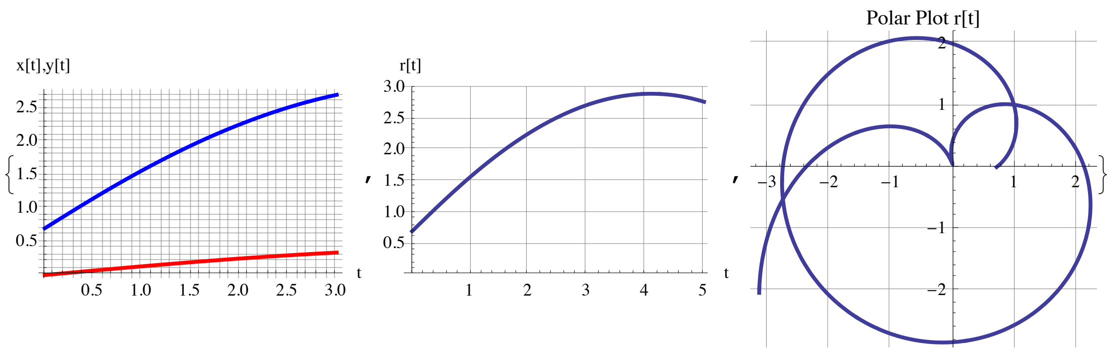

Figure 2. The force strength is $\xi = 2.25$, and initial values are $\{r(0), \dot{r}(0), \theta(0), \dot{\theta}(0)\} = \{0.7, 0.91, 0, 0.19\}$

Description of the individual panel is the same as in the previous examples. It is worthwhile noting unlike the previous examples the orbital is not stable. The character of the pure radial force makes the particle orbiting about the center and then wandering away, tracing a non-returnable trajectory. For a better descriptive word maybe in this case "orbital" should be called "trajectory."

Example 3. Here we consider an example associated with the cell_{14}. We equate $\frac{1}{m} f_{14} \equiv -\xi\, r(t)\, \theta(t)$; where ξ is the force constant. The spirit of this theoretical suggested force is similar to the previous example. Meaning, no such force has been observed in nature, yet! However, as mentioned before, the analysis paves the road for the "what-if scenarios." Similar to the previous examples the angular momentum of the particle is conserved. For a set of parameters specified in the figure caption of Fig 3 the graphic information includes the orbital.

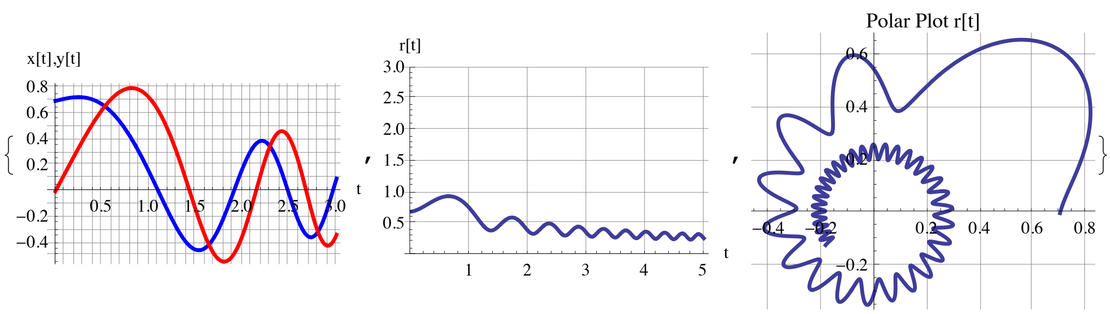

Figure 3. The force strength is $\xi = 4.5$, and initial values are $\{r(0), \dot{r}(0), \theta(0), \dot{\theta}(0)\} = \{0.7, 0.18, 0, 1.96\}$.

According to the plot of the middle panel the radial distance of the particle is a diminishing oscillatory function with respect to time, t. Its polar plot shown in the right panel is an interesting unstable orbital. Here the particle interestingly orbits about the center and contrary to example 2 stays in sight.

Example 4. As a last example we consider a case associated with cell_{32}. We equate $\frac{1}{m} f_{32} \equiv -\xi\, r(t)$ and $\frac{1}{m} g_{32} \equiv -\xi\, \theta(t)$; here ξ is the force constant. Our approach is quite general such that one may even adjust the individual force strength ξ along the radial and the azimuthal directions. This example, unlike the previous examples, doesn't conserve the angular momentum. For a set of parameters specified in the figure caption of Fig 4 the graphic information includes the orbital.

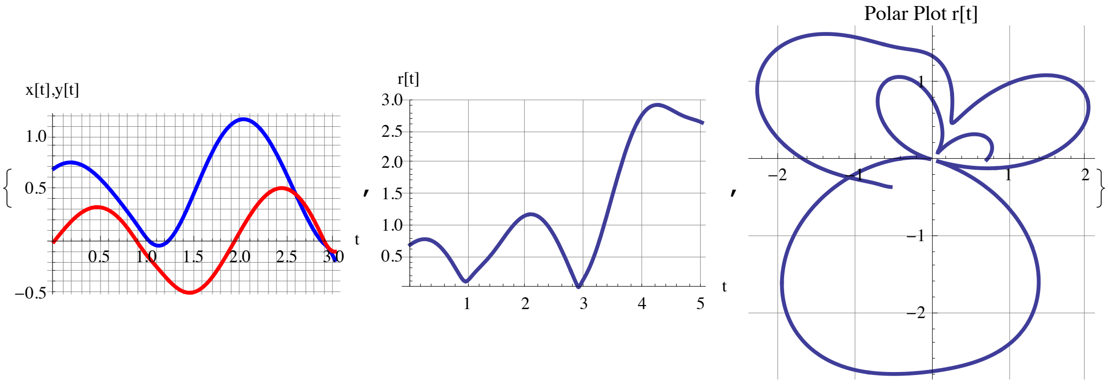

Figure 4. The common force strength is $\xi = 5.15$, and initial values are $\{r(0), \dot{r}(0), \theta(0), \dot{\theta}(0)\} = \{0.7, 0.65, 0, 1.61\}$.

The orbital shown in the right panel is somewhat interesting. The rest of the plots are self explanatory.

Conclusions

Motion of a particle under the influence of conventional forces such as gravity and electrostatic are known [3,4 and 7]. These forces are purely radial and do conserve the angular momentum. The corresponding equations of motion are solvable analytic differential equations. The orbitals are stable conic sections. Advances in CAS make it possible to augment the area of the study. As such, our current work investigates the impact of generalized forces on the orbitals. Our work even augments our previous work [1,2]. We have introduced fifteen classes of forces. We show the details of only a handful of cases. However, according to what we have introduced in the text our approach recognizes no limitation. As we mentioned in the Motivation and Goals section and in the body of the text of example 2, the freedom of choosing the force functions in any of the fifteen classes puts no limit on our theoretical investigation. To the author's knowledge, there are no other comparably investigated reports in this area.
Our approach is purely numeric and is based on numeric solution of differential equations provided by *Mathematica*. We include a *Mathematica* code so that the interested reader is able to apply the code investigating the orbitals/trajectories of a particular force.

Appendix. *Mathematica* code. Here is the code that runs Example 4 in the text. In this example the right hand sides of (3) are $-\xi\, r(t)$ and $-\xi\, \theta(t)$, respectively. However, the given code can run for any desired case. To do so in the first line of the code one needs to replace the right hand side of the corresponding equations with the appropriate functions.

```
Manipulate[
 solrφt =
  NDSolve[{r''[t] - r[t] φ'[t]^2 == -ξ r[t], 2.0 r'[t] φ'[t] + r[t] φ''[t] == -ξ φ[t],
    r[0] == α, r'[0] == β, φ[0] == 0, φ'[0] == γ}, {r[t], φ[t]}, {t, 0, 10}];
 {Plot[Evaluate[{r[t] Cos[φ[t]], r[t] Sin[φ[t]]} /. solrφt], {t, 0, 3},
   PlotStyle → {{Blue, Thick}, {Red, Thick}}, AxesLabel → {"t", "x[t],y[t]"},
   GridLines → {Range[0, 3, 0.1], Range[-1, 3, 0.1]}],
  Plot[r[t] /. solrφt, {t, 0, 5}, PlotRange → {0, 3}, PlotStyle → {Thick},
   GridLines → Automatic, AxesLabel → {"t", "r[t]"}],
  PolarPlot[r[t] /. solrφt, {t, 0, 10}, PlotLabel → "Polar Plot r[t]",
   PlotStyle → {Thick}, GridLines → Automatic]
 },
 {ξ, 0.5, 6.0, 0.05}, {{α, 0.7, "r[0]"}, 0.7, 1.2, 0.01}, {{β, 0, "r'[0]"}, 0, 1, 0.01},
 {{γ, -2, "φ'[0]"}, -2, 3, 0.01}, ControlPlacement → Left]
```

Acknowledgement

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