IMS 2015
12th International Mathematica Symposium

Method of Inverse Differential Operators

Analytic Solutions of 2nd Order PDEs with Initial Value and Boundary Condition Problems

Robert Kragler

Weingarten University of Applied Sciences
kragler@hs-weingarten.de
http://portal.hs-weingarten.de/web/kragler/mathematica

Abstract

The implementation of the **M**ethod of **I**nverse **D**ifferential **O**perators (**MIDO**) which is an extension to **DSolve** in *Mathematica* is applied to **I**nitial **V**alue **P**roblems (IVP) and **B**oundary **C**onditions (BC) of homogeneous and non-homogeneous, linear PDEs of 2nd order such as the Laplace equation, the wave equation and the heat/ diffusion equation with respect to different types of boundary conditions. For selected examples of 2nd order PDEs explicit analytical solutions will be given in order to demonstrate the potential of MIDO.

As to the homogeneous *Laplace equation* (with 3 or 4 spatial dimensions) the solutions will be obtained by quaternion factorization of the differential polynomial (e.g. in 3 variables x,y,z)

$$X_3 = \alpha^2\,\mathcal{D}_x{}^2 + \beta^2\,\mathcal{D}_y{}^2 + \gamma^2\,\mathcal{D}_z{}^2\,;$$

hence the homogenous PDE

$$\left(\mathcal{D}_x\,\alpha - \mathcal{D}_y\,\mathbf{i}_q\,\beta - \mathcal{D}_z\,\mathbf{j}_q\,\gamma\right)\left(\mathcal{D}_x\,\alpha + \mathcal{D}_y\,\mathbf{i}_q\,\beta + \mathcal{D}_z\,\mathbf{j}_q\,\gamma\right)u(x,\ y,\ z) = 0$$

will be factorized and solved.

In order to obtain the analytical solution of the *Wave equation* (in 1 spatial dimension) with a inhomogeneity such as $\left(\mathcal{D}_t{}^2 - c^2\,\mathcal{D}_x{}^2\right)u(x,\ t) = e^{-|x|}\,\sin(t)$ it is required to resort to distributions (for example replacing the built-in *Mathematica* function **Abs[x]** by an equivalent distribution **abs(x) = x (Θ(x) - Θ(-x))** where **Θ** is the built-in HeavisideTheta function).

With regards to the *Heat/Diffusion equation* $\left(\tau\,\mathcal{D}_t - \mathcal{D}_x{}^2 - \mathcal{D}_y{}^2\right)u(t,\ x,\ y) = 0$ analytical solutions are given for six different types of (non-)homogeneous boundary conditions and initial values for 1 spatial dimension. In the case of 2 spatial dimensions boundary conditions of Dirichlet and v. Neumann type are investigated.

Introduction

Implementation of MIDO : *Mathematica* package DESolve0.m

In order to do the calculation of the examples given here the the MIDO procedures have to be loaded first. **Needs** loads the *Mathematica package* (for version 9.0) **DESolve0.m** which comprises all definitions, procedures, replacement rules etc. required to run the essential procedures **DESolve** and **initialValueProblem**.

```
Clear["Global`*"]
SetDirectory[NotebookDirectory[]];
Get["DESolve0`"]
```

After successful execution of the code the current *Mathematica* version, date and time are shown using **VersionDateTime**.

```
VersionDateTime
```

```
Mathematica V9.0.1 for Microsoft Windows (64-bit) (January 25, 2013
date= November 23, 2014; time= 17:52h
```

Procedures for Boundary Conditions and Initial Value Problems

In order to treat the *initial value problem* of 2nd order PDEs such as the *Laplace* and *Wave equation* the package **DESolve.m** the following procedures **DESolve,initialValueProblem** and **testIVP** have to be used.

```
? DESolve initialValueProblem testIVP
```

In the next two sections the 2d *Laplace equation* $(\mathcal{D}_x{}^2 + \mathcal{D}_y{}^2)\, u(x, y) = \phi(x, y)$ and the 1d *Wave equation* $(\mathcal{D}_x{}^2 - c_0{}^2\, \mathcal{D}_t{}^2)\, u(x, t) = \phi(x, t)$ will be investigated in detail; the results obtained are tested with the procedure **testIVP[χ,ϕ,u0,initCond,onoff,opt]**.

As to the *heat/diffusion equation* $(\tau\, \mathcal{D}_t - \mathcal{D}_x{}^2)\, u(x, y) = g(x)$ situation is a little bit different; there an initial value problem together with boundary conditions has to be considered. There are different types of boundary conditions which are accounted for by subsequent procedures **BIVProblem, homogeneousHeatEqnSolution** and **steadyStateSolution**.

Initial Value Problems for 2nd order PDEs

Laplace Equation with various boundary conditions

Due to Gaussian factorization of the homogeneous *Laplace equation* (in 2 dimensions) such as $(\mathcal{D}_y + i\, \mathcal{D}_x)(\mathcal{D}_y - i\, \mathcal{D}_x)\, u(x, y) = 0$ the corresponding *solution* has the general form $u_h = f_{1,0}(x + i\, y) + f_{2,0}(x - i\, y)$.

```
χ = (𝒟x^2+ 𝒟y^2);
uh= DESolve[χ,0,"Off"]/.{x→ y1,y→ x1}/.{x1→ x,y1→y}
```

For higher dimensional Laplace equations (i.e. 3d or 4d) factorization is achieved with the option **QuaternionIntegers**; see details in Appendix 1)

In order to solve the Laplace equation $u_{xx} + u_{yy} = 0$ with *general* initial conditions $u(x, 0) = \phi(x)$, $u_y(x, 0) = \psi'(x)$ the method given in Part 1 will be applied.

Example 1: $(\mathcal{D}_y + i\,\mathcal{D}_x)(\mathcal{D}_y - i\,\mathcal{D}_x)\,u(x, y) = 0$ (*homogeneous Laplace equation*) with boundary conditions $\{\, u(x, 0) = e^{-x}\,, u_y(x, 0) = \frac{1}{x^2+1} \,\} \Rightarrow$

$$u_0\,(x, y) = \frac{1}{2}\,(e^{-x-i\,y}\,(1 + e^{2\,i\,y}) + i\arctan(x - i\,y) - i\arctan(x + i\,y)) = e^{-x}\cos(y) - \frac{1}{4}\log\left(\frac{x^2+(1-y)^2}{x^2+(1+y)^2}\right)$$

With the *general solution* $u(x, y) = f_1(x + i\,y) + f_2(x - i\,y)$ of the 2d Laplace equation and the identity between **ArcTan** and its representation through **Log** given in terms of a delayed rule :

```
arcTanRule:={(ArcTan[x_-ⅈ y_]-ArcTan[x_+ⅈ y_]):→ⅈ/2 Log[(x^2+(1-y)^2)/(x^2+(1+y)^2)]}
```

Solving the system of equations for $f_1(x)$ and $f_2(x)$ yields $f_{1,2}(x) = \frac{1}{2}\,(e^{-x} \mp i\arctan(x))$; applying the **arcTanRule** the final result is

```
u[x_,y_]:= f1[x+ⅈ y] + f2[x-ⅈ y]
initCond = {u[x,0] == e^-x, (∂y u[x,y]/.{y→ 0}) == 1/(x^2+1)};
u0= initialValueProblem[u[x,y],initCond,{x,y},"Off",fs]/.arcTanRule
```

initial conditions : $\left\{f_1[x] + f_2[x] == e^{-x},\ i\, f_1'[x] - i\, f_2'[x] == \frac{1}{1+x^2}\right\}$

$$\Longrightarrow u_0(x,y) = f_1[x + i\,y] + f_2[x - i\,y] = \frac{1}{2}\,i\,(\text{ArcTan}[x - i\,y] - \text{ArcTan}[x + i\,y]) + e^{-x}\,\text{Cos}[y]$$

$$e^{-x}\,\text{Cos}[y] - \frac{1}{4}\,\text{Log}\left[\frac{x^2 + (1-y)^2}{x^2 + (1+y)^2}\right]$$

Obviously, the solution u_h fulfills the Laplace equation together with the initial value conditions **initCond** :

```
testIVP[χ,0,u0,initCond,"Off",fs];
```

IVP solution $u_0 = e^{-x}\,\text{Cos}[y] - \frac{1}{4}\,\text{Log}\left[\frac{x^2 + (1-y)^2}{x^2 + (1+y)^2}\right]$

fulfills 2nd order PDE and initial|boundary conditions :

$\partial_{\{x,2\}}\#1 + \partial_{\{y,2\}}\#1\,\&\ @\ u[x, y]==0 \Longrightarrow$ **True**

$\left\{f_1[x] + f_2[x] == e^{-x},\ i\, f_1'[x] - i\, f_2'[x] == \frac{1}{1+x^2}\right\} \Longrightarrow$ **True**

Wave Equation with various initial values

For the homogeneous wave equation $(\mathcal{D}_t - c\,\mathcal{D}_x)(\mathcal{D}_t + c\,\mathcal{D}_x)\,u(x, t) = 0$ the d'Alembert solution $u_h = f_{1,0}[x + c\,t] + f_{2,0}[x - c\,t]$ is well known. $f_{1,0}$ and $f_{2,0}$ are arbitrary functions representing a left-/right-traveling wave. The velocity of light c is defined as a real, positive quantity. For the 3d plots of u_0 a scaling factor of 10^{-5} is used.

```
Clear[c];
$Assumptions = c ∈ Reals && c > 0;
Refine[c^2 > 0];
```

The following Example **2** for the *non-homogeneous 1d wave equation* is quite sophisticated and requires due to the factor $e^{-|x|}$ the distribution **abs[x]** instead of the built-in function **Abs[x]** (for a detailed discussion see Appendix 2).

Example 2 : $\left(\mathcal{D}_t^2 - c^2\,\mathcal{D}_x^2\right) u(x, t) = e^{-|x|}\sin(t)$ (*non-homogeneous wave equation*) with initial conditions $\{\,u(x, 0) = 0\,,\, u_t(x, 0) = e^{-|x|}\,\} \Rightarrow$

$$u_0(x, t) = -\frac{e^{-\text{abs}[x]}\sin(t)}{(1+c^2)} + \frac{(2+c^2)}{2c\,(1+c^2)}\left(e^{-\text{abs}[x-c\,t]}\,\Theta[+c\,t-x] + \left(2 - e^{-\text{abs}[x-c\,t]}\right)\Theta[-c\,t+x] + e^{-\text{abs}[x+c\,t]}\,\Theta[-c\,t-x] - \left(2 - e^{-\text{abs}[x+c\,t]}\right)\Theta[+c\,t+x]\right)$$

Here the *wave equation* $u_{t\,t} - c^2\,u_{x\,x} = e^{-|x|}\sin(t)$ together with the initial conditions : $u(x, 0) = 0,\ u_t(x, 0) = e^{-|x|}$ will be solved. The *particular solution* is given by :

$$u_p = \frac{1}{\left(\mathcal{D}_t^2 - c^2\,\mathcal{D}_x^2\right)}\,e^{-|x|}\sin(t) = -\frac{1}{1+c^2}\,e^{-|x|}\sin(t)$$

so that the *complete solution* is : $u(x, t) = f_{1,0}(x - c\,t) + f_{2,0}(x + c\,t) - \frac{1}{1+c^2}\,e^{-|x|}\sin(t)$.

To prove with **testDE** whether the *particular solution* $u_p = -\frac{1}{1+c^2}\,e^{-|x|}\sin(t)$ fulfills the non-homogeneous wave equation given above is not quite trivial due to $e^{-|x|}$.

```
χ = (𝒟t^2 - c^2 𝒟x^2);
ϕ := e^-Abs[x] Sin[t];
up = -(e^-Abs[x] Sin[t])/(1+c^2);
testDE[χ, ϕ, up, "Off"];
```

```
=================== Test of inhomogeneous PDE ====================
```

PDE : $-c^2\,u_{xx} + u_{tt} = e^{-\text{Abs}[x]}\,\text{Sin}[t]$

is of order 2 and has particular solution u_p : $-\frac{e^{-\text{Abs}[x]}\,\text{Sin}[t]}{1+c^2}$

satisfies PDE (True|False) ⟹ $\frac{c\,\text{Sin}[t]\left(-1 + \text{Abs}'[x]^2 - \text{Abs}''[x]\right)}{1+c^2} == 0$

Inspection of the *residual terms* **Sin[t]** $\left(-1 + \text{Abs}'[x]^2 - \text{Abs}''[x]\right)$ shows that for all real nonzero values of **x** the lhs should vanish because of $\partial_x \mathbf{Abs}[x] = \mathbf{sgn}(x)$ and $\partial_{xx} \mathbf{Abs}[x] = 2\,\delta(x)$. Thus the resulting factor $(-1 + \mathbf{sgn}\,(\mathbf{x})^2 - 2\,\delta\,(\mathbf{x}))$ would vanish. Here $\delta(\mathbf{x})$ is the **DiracDelta** function which, however, remains unevaluated for $\mathbf{x} = 0$.

Yet, if for the global variable **$simplify** (which is internally used within the procedure **testNonhomDE**) the assumption is made that
(i) the variable $\mathbf{x} \in \mathbf{R}$ is nonzero and (ii) the replacement $\{\mathbf{Abs}''[\mathbf{x}] \to 2\,\delta(x)\}$ is enforced then the correct result is obtained.

```
   $simplify := FullSimplify[#/.{Abs''[x] → 2δ(x)},{x∈R,x≠0}]&
testDE[χ,ϕ,up,"Off"];
   $simplify = fs;            (* reset to default *)
```

```
=================== Test of inhomogeneous PDE ====================

PDE : -c^2 uxx + utt= e^-Abs[x] Sin[t]

 is of order 2 and has particular solution up :   -(e^-Abs[x] Sin[t])/(1 + c^2)

  satisfies PDE (True|False) ⟹ True
```

Additional refinement :

The *particular* solution $u_p = -\frac{e^{-|x|}\sin(t)}{1+c^2}$ is redefined as $u_{p1} = -\frac{e^{-\mathrm{abs}[x]}\,\mathrm{Sin}[t]}{1+c^2}$ where **Abs[x]** is replaced by **abs[x]**.

```
up1:= - (e^-abs[x] Sin[t])/(1+c^2)
```

u_{p1} fulfills the non-homogeneous wave equation for all values $\mathbf{x} < 0$ and $\mathbf{x} > 0$ as can be shown by direct substitution of u_{p1}

```
(fs0[x] @ ((∂t,t#-c^2∂x,x#)& @ up1 == e^-abs[x] Sin[t])//Union)[[1]]
```

```
True
```

By means of the global variable **$subst ="True"** a substitution rule is switched on which replaces the piecewise function **pwFct** by an equivalent function (involving the function $\theta(\mathbf{x})$) $H\,(x) = e^{-\mathrm{abs}(x)}\,\theta(-x) + \left(2 - e^{-\mathrm{abs}(x)}\right)\theta(x)$ which is analytically simpler.

```
$subst = "True";
H[x_]:= ( HeavisideTheta[+x] (2-e^-abs[x]) +
          HeavisideTheta[-x] e^-abs[x]      )/. ΘRule
```

```
u[x_,t_]:= f1,0[x+c t] + f2,0[x-c t] - (e^-abs[x] Sin[t])/(1+c^2)/.ΘRule ;
initCond = { u[x,0]== 0,(∂t u[x,t]/.{t→0})== (e^-abs[x]/. ΘRule) };
u0i = initialValueProblem[ u[x,t],initCond,{x,t},"Off",sf];
```

initial conditions : $\left\{f_{1,0}[x] + f_{2,0}[x] == 0,\ -\dfrac{e^{-x\,(-\Theta[-x]+\Theta[x])}}{1+c^2} + c\, f_{1,0}'[x] - c\, f_{2,0}'[x] == e^{-x\,(-\Theta[-x]+\Theta[x])}\right\}$

Function obtained from integration solInt⟦2⟧ :
Integrate$\left[e^{x\,(\Theta[-x]-\Theta[x])}\right.$, x, Assumptions → x ∈ Reals$\left.\right]$ substituted by equivalent function
$H(x) = e^{-x\,(-\Theta[-x]+\Theta[x])}\,\Theta[-x] + \left(2 - e^{-x\,(-\Theta[-x]+\Theta[x])}\right)\Theta[x]$

$$\Longrightarrow u_0(x,t) = f_{1,0}[c\,t+x] + f_{2,0}[-c\,t+x] - \frac{e^{-x\,(-\Theta[-x]+\Theta[x])}\,\mathrm{Sin}[t]}{1+c^2}$$
$$= \frac{1}{2\,(1+c^2)}\left(-2\,e^{x\,(\Theta[-x]-\Theta[x])}\,\mathrm{Sin}[t] + \frac{(2+c^2)\,e^{(c\,t+x)\,(\Theta[-c\,t-x]-\Theta[c\,t+x])}\,\Theta[-c\,t-x]}{c} - \frac{(2+c^2)\,e^{(-c\,t+x)\,(\Theta[c\,t-x]-\Theta[-c\,t+x])}\,\Theta[c\,t-x]}{c} - \frac{2\,(2+c^2)\,\Theta[-c\,t+x]}{c} + \frac{(2+c^2)\,e^{(-c\,t+x)\,(\Theta[c\,t-x]-\Theta[-c\,t+x])}\,\Theta[-c\,t+x]}{c} + \frac{2\,(2+c^2)\,\Theta[c\,t+x]}{c} - \frac{(2+c^2)\,e^{(c\,t+x)\,(\Theta[-c\,t-x]-\Theta[c\,t+x])}\,\Theta[c\,t+x]}{c}\right)$$

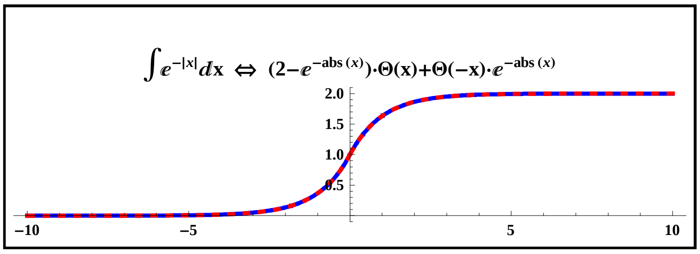
Note : integration of the *second initial condition* $u_t(x,0) = e^{-|x|}$ involves the integral $\int e^{-|x|}\,dx$ (with the representation **|*x*| ≡ Abs[*x*]**)

```
𝒥 := Integrate[#,x, Assumptions → x ∈ Reals]& ;
int = 𝒥 @ ((-e^(-Abs[x])/(1+c^2) + 2c f1,0′[x] == e^(-Abs[x]) //Solve[#,{(f1,0)′[x]}]&)/.
                                  {(2+c^2)/(2c (1+c^2)) → β}/.R2L//Flatten);
int⟦1⟧ → β(int⟦2⟧/β //sf)
```

$f_{1,0}[x] \to \beta\left(\begin{cases} e^{x} & x \le 0 \\ 2 - e^{-x} & \text{True}\end{cases}\right)$

which gives rise to a *piecewise function* **pwFct** $= \begin{cases} +e^{+x} & x < 0 \\ 2 - e^{-x} & \text{True}\end{cases}$. This function is substituted by an equivalent expression $H(x) = e^{-\mathrm{abs}[x]}\,\theta(-x) + \left(2 - e^{-\mathrm{abs}[x]}\right)\theta(x)$ which is analytically simpler to handle. Here, $\theta(x)$ denotes the distribution **Heaviside-Theta**.

The following plot shows that both function representations are equivalent. The blue curve is the result of the integration of $\int e^{-|x|}\,dx$, the dashed red curve shows the equivalent representation $\left(2 - e^{-\mathrm{abs}[x]}\right)\Theta[x] + \Theta[-x]\,e^{-\mathrm{abs}[x]}$.



With the replacement of the piecewise function **pwFct** by **H[x]** the final result can be casted into an even more transparent shape :

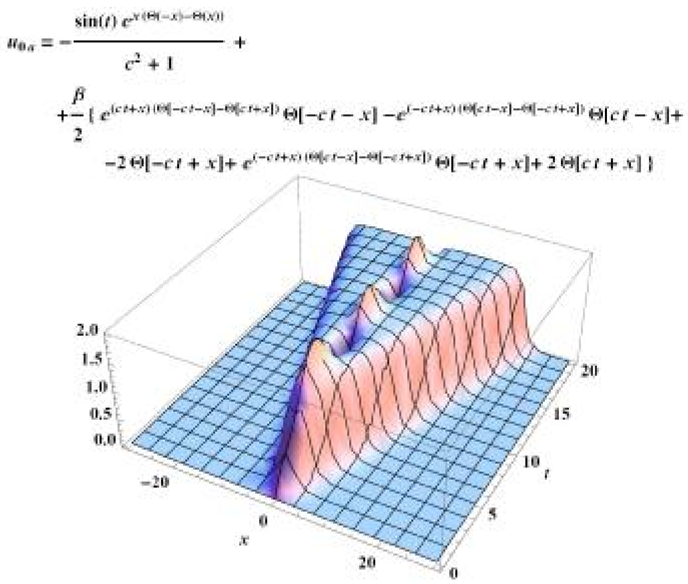
$$\mathbf{u_0\,(x,\ t)} = -\frac{e^{-|x|}\sin(t)}{(1+c^2)} + \frac{(2+c^2)}{2\,c\,(1+c^2)}\left(e^{-|x-c\,t|}\,\Theta[+c\,t-x] + \left(2-e^{-|x-c\,t|}\right)\Theta[-c\,t+x] + e^{-|x+c\,t|}\,\Theta[-c\,t-x] - \left(2-e^{-|x+c\,t|}\right)\Theta[+c\,t+x]\right)$$

with $|\mathbf{x}| = \mathbf{x}\ (\Theta[-\mathbf{x}] - \Theta[\mathbf{x}])$ and common factor $\beta = \frac{(2+c^2)}{c\,(1+c^2)}$ in front of the final result.

```
(u0β = u0i //βCollect)
```

$$-\frac{e^{x\,(\Theta[-x]-\Theta[x])}\,\mathrm{Sin}[t]}{1+c^2} + \frac{1}{2}\beta\left(e^{(c\,t+x)\,(\Theta[-c\,t-x]-\Theta[c\,t+x])}\,\Theta[-c\,t-x] - e^{(-c\,t+x)\,(\Theta[c\,t-x]-\Theta[-c\,t+x])}\,\Theta[c\,t-x] - 2\,\Theta[-c\,t+x] + e^{(-c\,t+x)\,(\Theta[c\,t-x]-\Theta[-c\,t+x])}\,\Theta[-c\,t+x] + 2\,\Theta[c\,t+x] - e^{(c\,t+x)\,(\Theta[-c\,t-x]-\Theta[c\,t+x])}\,\Theta[c\,t+x]\right)$$

The 3d-plot of $\boldsymbol{u_0(x,\ t)}$ has the appearance :



For a more sophisticated treatment of an initial value problem there are besides **`FullSimplify`** several simplification procedures (which distinguish cases $\mathbf{x} < \mathbf{0}$ and $\mathbf{x} > \mathbf{0}$) and replacement rules for expressions involving the **`HeavisideTheta`** function needed which are provided in the list **`optF`**. The global variable **`$refine = "True"`** (default value is **`"False"`**) switches within the procedure **`testIVP`** to cope for this situation.

Further details are suppressed but incorporated in the function **`testIVP`**.

See the simpler example 2a

Heat/Diffusion Equation with various boundary conditions and initial values

The *homogeneous* heat/diffusion equation (in principle, for any spatial dimension) is solvable by *separation of variables.* This is achieved by

```
? homogeneousHeatEqnSolution
```

To account for different boundary conditions and initial values there is the procedure **`BIVProblem`** available. It provides for the 1-dimensional heat equation several types of boundary conditions (parameter **BC = 1,...6**) which are described below and will be illustrated in several examples which are taken from [4,5]. The boundary conditions with **BC=7, 8** deal with the 2-dim. heat/diffusion equation.

```
? BIVProblem
```

1-dimensional heat/diffusion equation

In the case of one spatial variable, say **x**, substitution of $u\,(t,\,x) = T(t)\cdot X(x)$ into $(\tau\,\mathcal{D}_t - \mathcal{D}_{xx})\,u(t,\,x) = 0$ separates the PDE $\frac{\tau\,\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$ into two ODEs of 1st and 2nd order $\tau\,\dot{T}(t) + \lambda^2\,T(t) = 0$ and $X''(x) + \lambda^2\,X(x) = 0$. While the ODE for the variable t has the solution $T(t) = c_0\,e^{-\lambda^2\,t/\tau}$, the solution of the second ODE for spatial variable x is $X\,(x) = c_1\,\sin(\lambda\,x) + c_2\cos\,(\lambda\,x)$. τ is the relaxation time, λ the eigenvalue.

The coefficients c_0, c_1 and c_2 and the eigenvalue λ will be determined through boundary conditions. With the homogeneous boundary conditions $X\,(0) = 0 = X(L_1)$ the value for $\lambda(n) = \frac{n\,\pi}{L_1}$ (**n = 0,1,...**) is obtained.

```
χ = (τ 𝒟t - 𝒟x^2);
uh= homogeneousHeatEqnSolution [χ,"On",sf]
```

Case (1)

The boundary conditions are called *homogeneous* if the solution $u(t,\,x)$ fulfills the following boundary conditions: $u\,(t,\,x = L_0) = u(t,\,x = L_1) = T_0$. The initial condition is given by $u\,(0,\,x) = F(x)$ for ($L_0 \le x \le L_1$).

Example 3.1 : $\tau = 30,\ \{T_0\,,\,T_1\} = \{0,\,0\},\ \{L_0,\,L_1\} = \{0,\,1\},\ f(x) = \theta\left(\frac{3}{5} - x\right)\theta\left(x - \frac{2}{5}\right),$
$g = 0,\ u(t,\,0) = u(t,\,1) = 0\ ,\ u(0,\,x) = f(x)$ for $(0 < x < 1,\ 0 \le t)$

```
Clear[f,g]; τ = 30.;
χ =(τ 𝒟t - 𝒟x^2);
{T0,T1}= {0,0}; {L0,L1}= {0,1};
g[x_]:= 0;
f[x_]:= UnitStep[x-2/5] UnitStep[3/5-x];
u51= BIVProblem[χ,BC=1,{T0,T1},{L0,L1},f[x],0,51,"On",sf];
```

apply homogeneous boundary conditions : $u(t,L_0) = u(t,L_1) = 0$
where $\{L_0,L_1\} = \{0,\,1\};\ \{T_0,T_1\} = \{0,\,0\}$
and initial condition : $u(t=0,x)\ \Longrightarrow\ F(x) = \theta\left(\frac{3}{5} - x\right)\theta\left(x - \frac{2}{5}\right)$

$\Longrightarrow X_n(x) = \mathrm{Sin}[n\,\pi\,x]$ for boundary values $\{X(0) = 0,\ X(1) = 0\}$
$\Longrightarrow T_n(t) = e^{-0.328987\,n^2\,t}\,\gamma[n]$ eigenvalues $\lambda_n = (n\,\pi)$

Fourier coefficient: $\Longrightarrow \gamma_n = \frac{2}{L}\int_0^1 F(x)\cdot X_n(x)\,dx = \frac{2\left(\mathrm{Cos}\left[\frac{2\,n\,\pi}{5}\right] - \mathrm{Cos}\left[\frac{3\,n\,\pi}{5}\right]\right)}{n\,\pi}$

with $F(x) = \theta\left(\frac{3}{5} - x\right)\theta\left(x - \frac{2}{5}\right)$

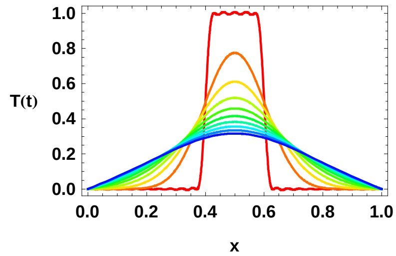
Fourier coefficients $\gamma_n = \dfrac{2\left(\text{Cos}\left[\frac{2 n \pi}{5}\right] - \text{Cos}\left[\frac{3 n \pi}{5}\right]\right)}{n \pi}$

$n = 2k-1$ (odd) $\Longrightarrow \gamma_n = \dfrac{2\left(\text{Cos}\left[\frac{2}{5}(-1+2k)\pi\right] + \text{Cos}\left[\frac{2}{5}(\pi + 3k\pi)\right]\right)}{(-1+2k)\pi}$

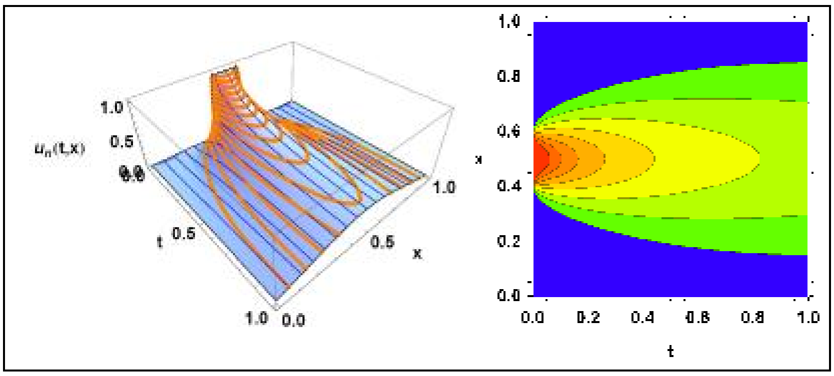
$n = 2k$ (even) $\Longrightarrow \gamma_n = 0$

$$\Longrightarrow u_\infty(t, x) = \sum_{n=1}^{\infty} \frac{1}{n\pi} 2\, e^{-0.328987\, n^2 t} \left(\text{Cos}\left[\frac{2 n \pi}{5}\right] - \text{Cos}\left[\frac{3 n \pi}{5}\right]\right) \text{Sin}[n \pi x]$$

```
showTimeSteps[u51/.{x → ξ,t → θ},{θ,2*10^-3,1,.1},{ξ,0,1},11]
```




```
showGraph[ u51/.{x → ξ,t → θ},{θ,10^-3,1},{ξ,0,1}, ViewPoint → {1,-0.9,1.0}]
```



Case (2)

The boundary conditions are called *non-homogeneous* if $u(t, x = L_0) = T_0$, $u(t, x = L_1) = T_1$ (with $T_0 \neq T_1$) which require a modification of the treatment in example 3 in order to introduce homogeneous boundary conditions to the problem. Therefore, the solution $u(t, x) = s(x) + v(t, x)$ is split into a (time-independent) *steady-state solution* $s(x) = \lim_{t \to \infty} u(t, x)$ and another part $v(t, x)$ which is called the *transition temperature*.

```
? steadyStateSolution
```

Substitution of $u(t, x)$ into the PDE leads to two differential equations :

(i) the steady-state equation $s''(x) == 0$ with $s(L_0) = T_0$ and $s(L_1) = T_1$ and

(ii) the heat equation $(\tau \mathcal{D}_t - \mathcal{D}_{xx})\, v(t, x) = 0$ now with homogeneous boundary conditions $v(t, L_0) = v(t, L_1) = 0$ for $t > 0$ and $v(0, x) = f(x) - s(x)$ for $L_0 < x < L_1$.

Accounting for the initial temperature T_0 one has $u(0, x) = v(0, x) + s(x) = f(x)$ so that the initial condition for $v(t, x)$ is given by $v(0, x) = f(x) - s(x)$.

In a *first step*, because $s(x)$ is needed in the calculation for $v(t, x)$, the steady-state solution is evaluated using **steadyStateSolution**; one finds $s(x) = T_0 + \frac{(T_1 - T_0)}{L} x$.

In a *second step* the heat equation for $v(t, x)$ is solved with the help of the procedure **homogeneousHeatEqnSolution** with homogeneous boundary conditions $\{T_0, T_1\} = \{0, 0\}$ for $v(t, x)$. However, instead of $f(x) = T_0$ to be used for the integral calculating γ_n the initial temperature is determined by $v(0, x) = f(x) - s(x) = -\frac{(T_1 - T_0)}{L_1} x$.

Example 3.2 : $\tau = 1, \{T_0, T_1\} = \{1, 10\}, \{L_0, L_1\} = \{0, 1\}, f(x) = (T_1 - T_0)\,\theta(x)\,\theta(1 - x),$
$g = 0,\ u(t, L_0) = T_0,\ u(t, L_1) = T_1,\ u(0, x) = f(x)$ for $(0 < x < 1, 0 \le t)$

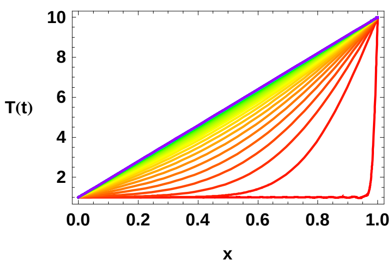
```
Clear[f,g]; τ = 1;
χ = (τ 𝒟t - 𝒟x^2);
{T0,T1}={1,10}; {L0,L1}={0,1};
g[x_]:= 0;
f[x_]:= (T1-T0)*UnitStep[x-L0]UnitStep[L1-x];
u51= BIVProblem[χ,BC=2,{T0,T1},{L0,L1},f[x],0,51,"Off",Short];
```

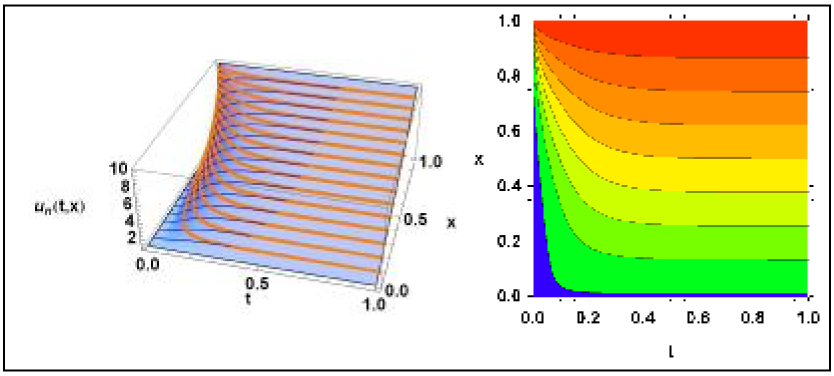
```
apply inhomogeneous boundary conditions :u(t,L0)=T0=1, u(t,L1)=T1=10
where {L0,L1}= {0, 1}; {T0,T1}= {1, 10}
and initial condition : u(t=0,x) ⟹ F(x)= 9 θ(1 - x) θ(x)

from steady-state solution :
initial transient temperature with {T0,T1}= {1, 10}
v(0,x)=9 θ(1 - x) θ(x)-s(x)= -9 x
```

$$\Longrightarrow u_\infty(t, x) = \sum_{n=1}^{\infty} \frac{18\,(-1)^n\, e^{-n^2 \pi^2 t}\, \mathrm{Sin}[n\,\pi\, x]}{n\,\pi}$$

The approximation of $u(t, x, \kappa)$ up to $\kappa = 51$ is plotted as a 3d plot and a contour plot.





Case (3)

In case of an additional ***source term*** $g(x)$ with $(\tau\,\mathcal{D}_t - \mathcal{D}_{xx})\,u(t,\,x) = g(x)$ again the ansatz $u(t,\,x) = s(x) + v(t,\,x)$ is made, however, $s(x)$ is the solution of the following non-homogeneous ODE $s''(x) = -g(x)$ for $(L_0 < x < L_1)$ with homogeneous boundary conditions $s(L_0) = 0 = s(L_1)$. Hence, $v(t,\,x)$ satisfies the homogeneous heat equation $(\tau\,\mathcal{D}_t - \mathcal{D}_{xx})\,v(t,\,x) = 0$ with boundary conditions $v(t,\,L_0) = 0 = v(t,\,L_1)$ for $t > 0$ and initial value $v(0,\,x) = -s(x)$.

Example 3.3 : $\tau = 1,\ \{T_0,\,T_1\} = \{0,\,1\},\ \{L_0,\,L_1\} = \{0,\,L\},\ g(x) = 17\cos\left(\frac{\pi}{2L}\,x\right),$
$f = 0,\ u(t,\,L_0) = T_0,\ u(t,\,L_1) = T_1,\ u(0,\,x) = 0$ for $(0 < x < L,\ 0 \le t)$

```
Clear[f,g];  τ = 1; L=. ;
χ = (τ 𝒟t - 𝒟x^2);
{T0,T1}= {0,1}; {L0,L1}= {0,L};
g[x_]:= 17 Cos[π/(2L) x];
steadyStateSolution[χ,{T0,T1},{L0,L1},g[x],"Off",sf]
```

$$\frac{-68\,L^3 + 68\,L^2\,x + \pi^2\,x + 68\,L^3\,\mathrm{Cos}\left[\frac{\pi\,x}{2\,L}\right]}{L\,\pi^2}$$

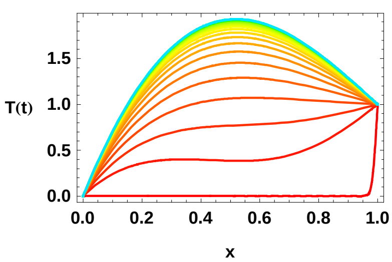
```
Clear[f,g]; τ = 1; L=. ;
χ = (τ 𝒟t - 𝒟x^2);
{T0,T1}= {0,1}; {L0,L1}= {0,L};
$Assumptions = L ∈ Reals;
g[x_]:= 17 Cos[π/(2L) x];
f[x_]:= 0;
u51= BIVProblem[χ,BC=3,{T0,T1},{L0,L1},0,g[x],51,"Off",Short];
```

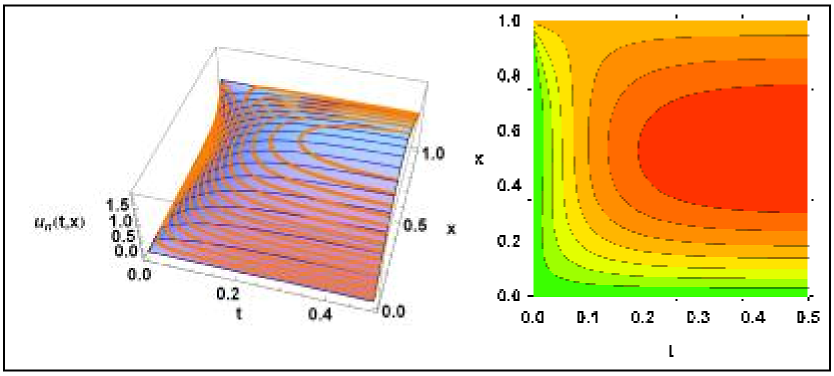
initial transient temperature with $\{T_0,T_1\} = \{0,\,1\}$

$$v(0,x) = 0 - s(x) = -\frac{-68\,L^3 + 68\,L^2\,x + \pi^2\,x + 68\,L^3\,\mathrm{Cos}\left[\frac{\pi\,x}{2\,L}\right]}{L\,\pi^2}$$

$$\Longrightarrow\ u_\infty(t,x) = \sum_{n=1}^{\infty} \frac{2\,e^{-\frac{n^2\pi^2 t}{L^2}}\left(-68\,L^2 + (-1)^n\left(-1 + 4\,n^2\right)\pi^2\right)\mathrm{Sin}\left[\frac{n\,\pi\,x}{L}\right]}{n\left(-1 + 4\,n^2\right)\pi^3}$$

The first $\kappa = 51$ terms of the eigenfunction expansion for the solution $u_\kappa(t,\,x)$ are displayed. The following plot shows the approximate solution at time steps of $t = 0.00,\ 0.03,\ \ldots\ 0.99$





Case (4)

In this example the rod has *isolated ends*. Thus, heat flow along the rod is zero at both ends $x = L_0$ and $x = L_1$. Mathematically, this kind of boundary condition are expressed as $u_x(t,\ 0) = u_x(t,\ L) = 0$ so that the following initial value and boundary condition problem must be solved : $u_x\ (t,\ L_0) = T_0 = u_x\ (t,\ L_1)$ and $u\ (0,\ x) = f(x)$ for $(L_0 < x < L_1)$. Therefore, the eigenvalue problem $X''(x) + \lambda^2\ X(x) = 0$ together with the boundary condition $\{\ X'\ (L_0\ ) == T_0,\ \ X'\ (L_1) == T_0\ \}$ has to be solved.

Example 3.4 : $\tau = 2,\ \{T_0\ ,\ T_1\} = \{0,\ 0\},\ \ \{L_0,\ L_1\} = \{0,\ 1\},\ \ f(x) = x^2,\ g = 0,$
$u_x(t,\ L_0) = T_0 = u_x\ (t,\ L_1),\ u\ (0,\ x) = f(x)$ for $(0 < x < L,\ \ 0 \le t)$

```
Clear[f,g]; τ= 2;
χ = (τ 𝒟t - 𝒟x²);
{T0,T1}= {0,0}; {L0,L1}= {0,1};
f[x_]:= x²;
g[x_]:= 0;
u51= BIVProblem[χ,BC=4,{T0,T1},{L0,L1},f[x],0,51,"Off", Shallow[#,{7,5}]& ];
```

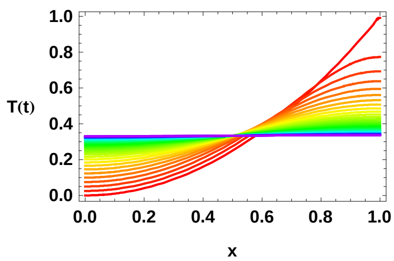
derivative boundary conditions: $\Longrightarrow$ eqn2X: $\{\lambda^2 X[x] + X''[x] == 0,\ X'[0] == 0\}$

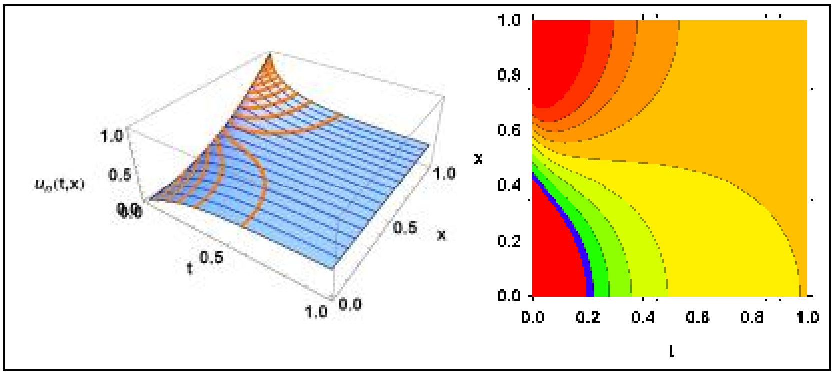
apply homogeneous boundary conditions : $X'(L_0) = X'(L_1) = 0$
where $\{L_0, L_1\} = \{0,\ 1\}$; $\{T_0, T_1\} = \{0,\ 0\}$
and initial condition : $u(t=0, x) \Longrightarrow F(x) = x^2$

Fourier coefficients $\gamma_n = \dfrac{4\ (-1)^n}{n^2\ \pi^2}$

n=2k-1 (odd) $\Longrightarrow \gamma_n = -\dfrac{4\ (-1)^{2k}}{(1 - 2\ k)^2\ \pi^2}$ n=2k (even) $\Longrightarrow \gamma_n = \dfrac{(-1)^{2k}}{k^2\ \pi^2}$

$$\Longrightarrow u_\infty(t, x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4\ (-1)^n\ e^{-\frac{1}{2} n^2 \pi^2 t}\ \mathrm{Cos}[n\ \pi\ x]}{n^2\ \pi^2}$$





Case (5)

In this case there are *mixed boundary conditions* : $u(t, L_0) = u_x(t, L_1) = T_0$ with $u(0, x) = f(x)$ for $(L_0 < x < L_1)$. Thus, the associated eigenvalue problem for the ODE $X''(x) + \lambda^2 X(x) == 0$ with boundary conditions $\{X(L_0) == T_0, X'(L_1) == T_0\}$ has to be solved.

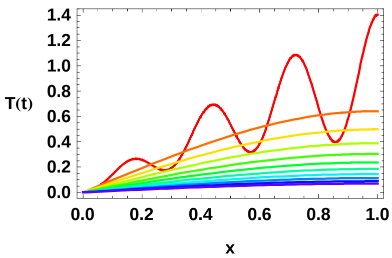
Example 3.5 : $\tau = 1$, $\{T_0, T_1\} = \{0, 0\}$, $\{L_0, L_1\} = \{0, L\}$, $f(x) = x \sin^2\left(\frac{7\pi}{2} x\right)$,
$g = 0$, $u(t, L_0) = T_0 = u_x(t, L_1)$, $u(0, x) = f(x)$ for $(L_0 < x < L_1, 0 \le t)$

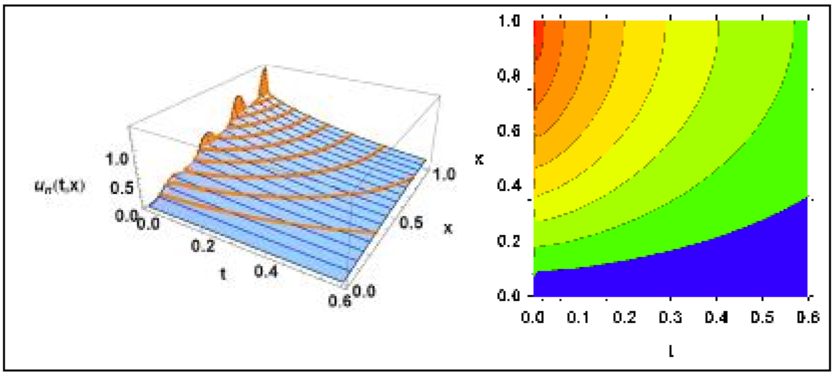
```
Clear[f,g];τ= 1;  L=. ;
χ = (τ 𝒟t - 𝒟x^2);
{T0,T1}= {0,0}; {L0,L1}= {0,L};
$Assumptions = (L ∈ Reals) && (L > 0);
f[x_]:= x Sin[7π/2 x]^2;
g[x_]:=0;
u51= BIVProblem[χ,BC=5,{T0,T1},{L0,L1},f[x],0,51,"Off",Shallow[#,{7,5}]& ];
```

apply homogeneous boundary conditions : $X(L_0) = X'(L_1) = 0 \implies X'(L_1): c_2\,\lambda\,\mathrm{Cos}[L\,\lambda] == 0$
where $\{L_0, L_1\} = \{0, L\}$; $\{T_0, T_1\} = \{0, 0\}$

with initial condition : $u(t=0, x) \implies F(x) = x \sin^2\left(\frac{7\pi x}{2}\right)$

$$\implies u_\infty(t,x) = \sum_{n=0}^{\infty} \frac{1}{\pi^2} 4 (-1)^n e^{-\frac{(-1+2n)^2 \pi^2 t}{4L^2}} L \left(-\frac{1}{(1-2n)^2} + \left(\left(196 L^2 + (1-2n)^2\right) \mathrm{Cos}[7 L \pi] + 7 L \left(196 L^2 - (1-2n)^2\right) \pi \,\mathrm{Sin}[7 L \pi]\right) \Big/ \left(-196 L^2 + (1-2n)^2\right)^2\right) \mathrm{Sin}\left[\frac{(-1+2n)\pi x}{2L}\right]$$





Case (6)

The boundary conditions are of *Robin* type at $x = L_0$ i.e. $u(t, L_0) - u_x(t, L_0) = T_0$ and of *v. Neumann* type at $x = L_1$ i.e. $u_x(t, L_1) = T_0$; the initial value is $u(0, x) = f(x)$. Therefore, the eigenvalue problem for $X''(x) + \lambda^2 X(x) = 0$ with boundary conditions $\{X(L_0) - X'(L_0) == T_0, X'(L) == T_1\}$ must be solved.

Example 3.6 : $\tau = 10$, $\{T_0, T_1\} = \{0, 0\}$, $\{L_0, L_1\} = \{0, 1\}$, $g = 0$,

$$f(x) = \frac{1}{4}\left(x^8\,\theta(1-x) + (1-x)^8\,\theta(x)\right) - 0.002,$$

$$u(t, L_0) = T_0,\ u(t, L_1) = T_1,\ u(0, x) = 0 \ \text{ for } (L_0 < x < L_1,\ 0 \le t)$$

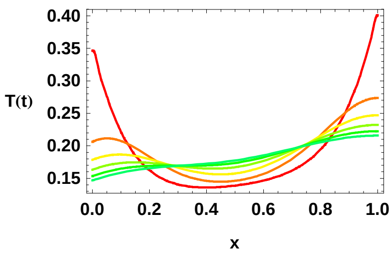
```
Clear[f,g]; τ= 10;
χ = (τ 𝒟t - 𝒟x^2);
{T0,T1}= {0,0}; {L0,L1}= {0,1};
f[x_]:= 1/4((1-x)^8*UnitStep[x]+ x^8*UnitStep[1-x])- 0.002;
g[x_]:= 0;                                              (* ⌚ *)
u51= BIVProblem[χ,BC=6,{T0,T1},{L0,L1},f[x],0,51,"Off",Short[#,4]& ];
```

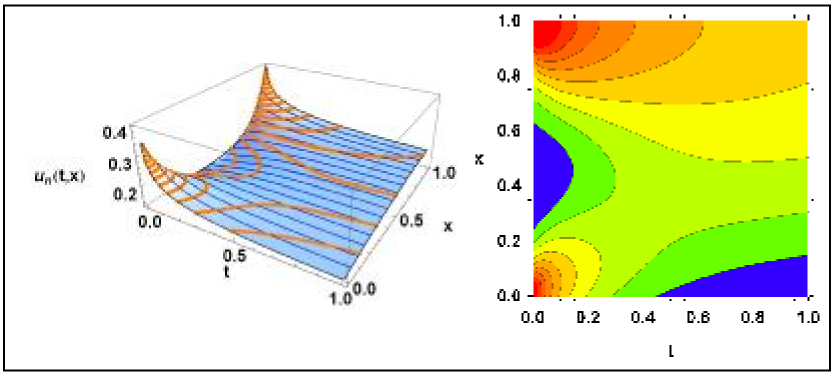
```
Robin type boundary conditions at x=L0:  ⟹ eqn2X: {λ^2 X[x] + X''[x] == 0, X[0] - X'[0] == 0}

apply homogeneous boundary conditions : X(L0)- X'(L0)= X'(L1)= 0

with initial condition : X(t=0,x) ⟹ F(x)= 1/4 (x^8 θ(1 - x) + (1 - x)^8 θ(x)) - 0.002
```

```
approximate solution returned from BIVProblem1 ⟹ n= 1...51
u51(t,x)= 0.162631 e^(-0.0740174 t) (0.860334 Cos[0.860334 x] + Sin[0.860334 x]) +
  <<49>> + 2.18802×10^-6 e^(-2467.6 t) (157.086 Cos[157.086 x] + Sin[157.086 x])
```





2-dimensional heat/diffusion equation

In the case of two spatial variables, say $(x,\ y)$, substituting now for the variables $(t,\ x,\ y)$ the ansatz for $u\,(t,\ x,\ y) = T\,(t)\cdot X\,(x)\cdot Y(y)$ into $\left(\tau\,\mathcal{D}_t - \mathcal{D}_{xx} - \mathcal{D}_{yy}\right) u(t,\ x,\ y) = 0$ separates this PDE into three ODEs : $\frac{\tau\,\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda^2$. The lhs depends on t, the rhs on (x,y) only; hence, each side is a constant (with respect to the variables occurring on the opposite side) which is called $-\lambda^2$. The same argument again applies to the rhs. Because the terms $\frac{X''(x)}{X(x)}$ and $\frac{Y''(y)}{Y(y)}$ are constants (with respect to the opposite side) they are called $-\mu_1{}^2$ and $-\mu_2{}^2$. Thus, there result three ODEs : $\tau\,\dot{T}(t) + \lambda^2\,T(t) = 0,\ \ X''(x) + \mu_1{}^2\,X(x) = 0$ and $Y''(y) + \mu_2{}^2\,Y(y) = 0$ with the constraint $\lambda^2 = \mu_1{}^2 + \mu_2{}^2$.

The coefficients $c_0,\ \ldots c_4$ together with the eigenvalues $\mu_1,\ \mu_2$ and λ are determined through the (homogeneous) boundary conditions $X\,(L_0) = 0 = X(L_1)$ and $Y\,(L_0) = 0 = Y(L_2)$. The values are for $\mu_1(n) = \frac{n\,\pi}{L_1}$, $\mu_2(m) = \frac{m\,\pi}{L_2}$ and finally $\lambda\,(n,\,m)^2 = \left(\frac{n\,\pi}{L_1}\right)^2 + \left(\frac{m\,\pi}{L_2}\right)^2$ (n,m = 0,1,2,...) .

```
χ = (τ 𝒟t - 𝒟x² - 𝒟y²);
uh= homogeneousHeatEqnSolution [χ,"Off",sf]
```

$c_0\, e^{-\frac{t\,\lambda^2}{\tau}}\,(c_1\,\mathrm{Cos}[x\,\mu 1] + c_2\,\mathrm{Sin}[x\,\mu 1])\,(c_3\,\mathrm{Cos}[y\,\mu 2] + c_4\,\mathrm{Sin}[y\,\mu 2])$

Subsequently, two types of boundary conditions will be investigated for the 2-dim heat/ diffusion equation :

(i) the *Dirichlet* (**BC=7**) and

(ii) the *v. Neumann* (**BC=8**) boundary conditions together with the initial value problem given by $u\,(t = 0,\ x,\ y) = f(x,\ y)$.

$$\tau\,\partial_t u - \Delta\,u = 0 \quad \text{for } (x,y) \in \mathcal{G} \text{ and } t > 0$$
$$\alpha\,u + (1-\alpha)\,\frac{\partial u}{\partial \bar{n}} = 0 \quad \text{for } (x,y) \in \partial\mathcal{G} \text{ and } t > 0 \text{ with } 0 \le \alpha \le 1$$
$$u(0,\ x,\ y) = f(x,\ y) \quad \text{for } (x,y) \in \overline{\mathcal{G}} = \mathcal{G} \cup \partial\mathcal{G}$$

$\mathcal{G} = \{(x,\ y) \mid L_0 < x < L_1\,,\ L_0 <\ y\,L_2\}$ denotes a rectangular domain with boundary $\partial\mathcal{G}$ as regards to the variables **(x,y)**. Here, $\Delta = \partial_{xx} + \partial_{yy}$ is the 2d Laplace operator, $\frac{\partial u}{\partial\,\hat{n}}$ the *normal derivative* of $u(t,\ x,\ y)$ at each point along the boundary $\partial\mathcal{G}$. It denotes the directional derivative of u in the direction of the unit vector $\hat{n}$ which is perpendicular to $\partial\mathcal{G}$ and points outwards with respect to the domain $\mathcal{G}$.

For simplicity a *rectangular domain* in **(x,y)** is investigated only where $\Delta = \partial_{xx} + \partial_{yy}$ is the 2d Laplace operator in Cartesian coordinates. For a *circular domain* the Laplace operator Δ must be expressed in polar coordinates (r,θ) : $\Delta = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \theta^2}\right)$ with $0 < r < R$ and $-\pi < \theta < \pi$ and the solution of the heat/ diffusion problem is given in terms of Bessel functions $J_m(r)$ **(m=0,1,2,...)**.

The following examples, however, consider rectangular domains only.

Case (7)

$\alpha = 1$ determines the *Dirichlet boundary conditions.* $\mathcal{G}$ is a rectangular domain with $L_0 < x < L_1,\ L_0 < y < L_2$. The initial value is $u(0, x, y) = f(x, y) = 2(\theta(x-2) - \theta(x-3))(\theta(y-2) - \theta(y-3))$ for $0 \le x,y \le 5$ and zero elsewhere on the boundary $\overline{\mathcal{G}}$.

$$\tau\, \partial_t u - \Delta u = 0 \quad \text{for } (x,y) \in \mathcal{G} \text{ and } t > 0$$
$$u(t, x, y) = 0 \quad \text{for } (x,y) \in \partial\mathcal{G} \text{ and } t > 0$$
$$u(0, x, y) = f(x, y) \quad \text{for } (x,y) \in \overline{\mathcal{G}} = \mathcal{G} \cup \partial\mathcal{G}$$

The ansatz $u(t, x, y) = T(t) \cdot X(x) \cdot Y(y)$ is made. Through the boundary conditions $X(L_0) = 0 = X(L_1)$ and $Y(L_0) = 0 = Y(L_2)$ the eigenvalues $\mu_1(m),\ \mu_2(n)$ as $\mu_1(m) = \frac{m\pi}{L_1}$ and $\mu_2(n) = \frac{n\pi}{L_2}$ for $X_m(x), Y_n(y, n)$ are determined with the constraint $\lambda(n, m)^2 = \left(\frac{m\pi}{L_1}\right)^2 + \left(\frac{n\pi}{L_2}\right)^2$ (m, n = 0,1,2, ...) for $T(t, m, n)$. The coefficients $\gamma_{m,n}$ are now defined by a *double Fourier sine series* representation of the initial condition $u(t=0, x, y) = f(x, y)$. Finally, the (approximate) solution is obtained in terms of a double summation (over **m** and **n**) of a Fourier sine series representation. Choosing **BC=7** the procedure **BIVProblem** accounts for the Dirichlet boundary conditions and the initial value problem described above.

Example 3.7 : $\tau = 10,\ \{T_x, T_y\} = \{0, 0\}, \{L_0, L_1, L_2\} = \{0, 5, 5\}, g(x, y) = 0,$
$f(x, y) = 2(\theta(x-2) - \theta(x-3))(\theta(y-2) - \theta(y-3))$,
$u(t, L_0, y) = T_x = u(t, L_1, y), u(t, x, L_0) = T_y = u(t, x, L_2)$
for $(L_0 < x < L_1, L_0 < y < L_2), 0 \le t)$

```
Clear[f,g]; τ = 10;
χ =(τ*𝒟t -𝒟x^2-𝒟y^2);
xyT= {0,0};
xyL= {L0,L1,L2}= {0,5,5};
f[x_,y_]:= 2(UnitStep[x-2]-UnitStep[x-3])*
            (UnitStep[y-2]-UnitStep[y-3]);
g[x_,y_]:= 0;                                  (* ⌚ 190 sec *)
u11= BIVProblem[χ,BC=7,xyT,xyL,f[x,y],0,11,"Off",sf];
```

```
Dirichlet boundary conditions for u(t,x,y)=T(t)X(x)Y(y) :
in rectangular domain  𝒢 = {(x,y)| (L0 < x < L1),(L0 < y < L2)}
with boundary ∂𝒢 = {(x,y)| for x={L0|L1}∧(L0≤y≤L2) and y={L0|L2}∧(L0≤x≤L1)}
where {L0,L1}= {0, 5} {L0,L2}= {0, 5} and  X(L0)=0=X(L1), Y(L0)=0=Y(L2) with
initial condition : u(t=0,x,y) ⟹ F(x,y)= 2 (θ(x - 2) - θ(x - 3)) (θ(y - 2) - θ(y - 3))
```

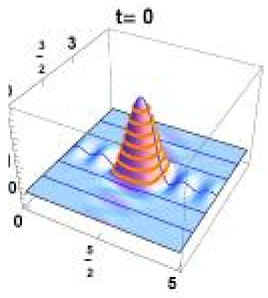
Fourier coefficient: $\gamma_{m,n} = \dfrac{4}{L1 \cdot L2}\displaystyle\int_{L0}^{L1}\int_{L0}^{L2} F(x,y) X_m(x) Y_n(y)\, dy\, dx =$

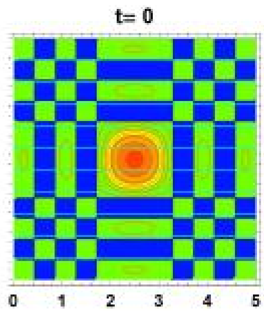
$$\frac{8\left(\cos\left(\frac{2\pi m}{5}\right) - \cos\left(\frac{3\pi m}{5}\right)\right)\left(\cos\left(\frac{2\pi n}{5}\right) - \cos\left(\frac{3\pi n}{5}\right)\right)}{\pi^2 m n}$$

with $F(x,y) = 2(\theta(x-2) - \theta(x-3))(\theta(y-2) - \theta(y-3))$

$$\Longrightarrow u_\infty(t,x,y) = \sum_{m,n=1}^{\infty} \frac{1}{\pi^2 m n} 8\left(\cos\left(\frac{2\pi m}{5}\right) - \cos\left(\frac{3\pi m}{5}\right)\right)\left(\cos\left(\frac{2\pi n}{5}\right) - \cos\left(\frac{3\pi n}{5}\right)\right) e^{-\frac{1}{250}\pi^2 t (m^2+n^2)} \sin\left(\frac{\pi m x}{5}\right) \sin\left(\frac{\pi n y}{5}\right)$$

```
{L0,L1,L2}= {0,5,5};
showAnimation[ u11/.{t → θ,x → ξ,y → η},
           {θ,0,5.,.05},{ξ,L0,L1},{η,L0,L2},PlotRange → {-.6,3}]
```





Case (8)

$\alpha = \mathbf{0}$ defines the *von Neumann boundary conditions*

$$\tau\,\partial_t u - \Delta u = 0 \quad \text{for } (x,y) \in \mathcal{G} \text{ and } t > 0$$
$$\frac{\partial u}{\partial \hat{n}} = 0 \quad \text{for } (x,y) \in \partial\mathcal{G} \text{ and } t > 0$$
$$u(0, x, y) = f(x, y) \quad \text{for } (x,y) \in \overline{\mathcal{G}} = \mathcal{G} \cup \partial\mathcal{G}$$

Again, a rectangular domain $\mathcal{G}$ is considered with $L_0 < x < L_1$, $L_0 < y < L_2$. Along each vertical edge (where $x = 0 \mid L_1$) the normal derivative is simply $\partial_x u$; along each horizontal edge (where $y = 0 \mid L_2$) the normal derivative is $\partial_y u$. Thus, for the problem under consideration the condition simply is $\partial_x u = 0$ for $x = L_0 \mid L_1$, $\partial_y u = 0$ for $y = L_0 \mid L_2$ $(t > 0)$ and initial condition $u(t = 0, x, y) = f(x, y)$ for $(x,y) \in \mathcal{G}$.

Example 3.8 : $\tau = 2$, $\{T_x, T_y\} = \{0, 0\}$, $\{L_0, L_1, L_2\} = \{0, 5, 3\}$, $g(x, y) = 0$,
$f(x, y) = 3\,\theta(x-4)\,\theta(y-2)$, $u(t, L_0, y) = T_x = u(t, L_1, y)$,
$u(t, x, L_0) = T_y = u(t, x, L_2)$ for $(L_0 < x < L_1, L_0 < y < L_2), 0 \le t)$

The initial condition is defined by the function $f(x, y)$

$$u(0, x, y) = f(x, y) = \begin{cases} 3\,\theta(x-4),\,\theta(y-2) & \text{if } 4 < x,\ 2 < y \\ 0 & \text{elsewhere in } \overline{\mathcal{G}} \end{cases}$$

```
Clear[f,g]; τ = 2;
χ =(τ 𝒟t - 𝒟x^2-𝒟y^2);
xyT= {0,0};
xyL= {L0,L1,L2}= {0,5,3};
f[x_,y_]:= 3 HeavisideTheta[x-4]HeavisideTheta[y-2];
g[x_,y_]:= 0;                            (* ⏱ 52 sec *)
u5= BIVProblem[χ,BC=8,xyT,xyL,f[x,y],0,5,"Off",sf];
```

v.Neumann boundary conditions for $u(t,x,y)=T(t)X(x)Y(y)$:
in rectangular domain $\mathcal{G} = \{(x,y) \mid (L_0 < x < L_1),(L_0 < y < L_2)\}$
with boundary $\partial\mathcal{G} = \{(x,y) \mid$ for $x=\{L_0|L_1\} \wedge (L_0 \le y \le L_2)$ and $y=\{L_0|L_2\} \wedge (L_0 \le x \le L_1)\}$
where $\{L_0,L_1\}=\{0, 5\}$ $\{L_0,L_2\}=\{0, 3\}$ and $X'(L_0)=0=X'(L_1)$, $Y'(L_0)=0=Y(L_2)$ with
initial condition : $u(t=0,x,y) \Longrightarrow F(x,y) = 3\,\theta(x-4)\,\theta(y-2)$

(1) v.Neumann boundary conditions for L_0 : $\{X'(L_0=0)=T_x,\ Y'(L_0=0)=T_y\}$ with $\{T_x,T_y\}=\{0, 0\}$
$\Longrightarrow X(x) = c_1 \mathrm{Cos}[x\,\mu 1];\ Y(y) = c_3 \mathrm{Cos}[y\,\mu 2]$

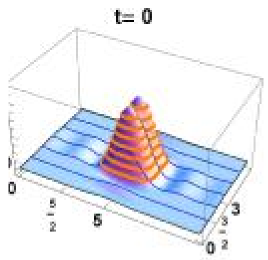
(2) v. Neumann boundary conditions for $L_{1,2}$:
$\{X'(L_1=5)=T_x,\ Y'(L_2=3)=T_y\}$ with $\{T_x,T_y\}=\{0, 0\}$
$$\Longrightarrow \{\mu_1(m),\mu_2(n)\} = \left\{\frac{m\pi}{l1}, \frac{n\pi}{l2}\right\} /.\{l_1\to 5, l_2\to 3\}$$

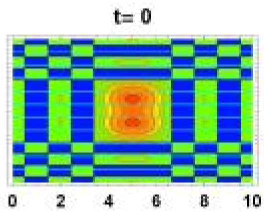
Fourier coefficient:
$$\Upsilon_{m,n} = \frac{4}{L1 \cdot L2}\int_{L0}^{L1}\int_{L0}^{L2} F(x,y)\,X_m(x)\,Y_n(y)\,dy\,dx = \frac{12\,\mathrm{Sin}\left[\frac{4m\pi}{5}\right]\mathrm{Sin}\left[\frac{2n\pi}{3}\right]}{m\,n\,\pi^2}$$

special values : $\Upsilon_{0,0} = \frac{1}{5}$; $\Upsilon_{m,0} = -\frac{2\,\mathrm{Sin}\left[\frac{4m\pi}{5}\right]}{m\pi}$; $\Upsilon_{0,n} = -\frac{6\,\mathrm{Sin}\left[\frac{2n\pi}{3}\right]}{5n\pi}$

with $F(x,y) = 3\,\theta(x-4)\,\theta(y-2)$ for $(x,y) \in \mathcal{G}$

$$\Longrightarrow u_\infty(t,x,y) = \sum_{m,n=0}^{\infty} \frac{12 \sin\left(\frac{4\pi m}{5}\right) \sin\left(\frac{2\pi n}{3}\right) e^{-\frac{1}{450}\pi^2 t\,(9m^2+25n^2)} \cos\left(\frac{\pi m x}{5}\right) \cos\left(\frac{\pi n y}{3}\right)}{\pi^2\, m\, n}$$





Results

Explicit analytical solutions for several well-known 2nd order PDEs were calculated. The PDEs which were considered are :

(1) the homogeneous 2d *Laplace equation* with different boundary conditions,

(2) the homogeneous / non-homogeneous 1d *wave equation* with different initial conditions

(3) the *1d- heat / diffusion equation* for several types of boundary conditions (BC) (such as homogeneous and non-homogeneous BC with/without a source term, isolated BC, mixed BC of Robin type) and

(4) the *2d heat / diffusion equation* for two types boundary conditions (i.e. Dirichlet and v. Neumann BC). The solutions in terms of Fourier sine/cosine series expansions are shown as 3d plots and animated contour plots.

Conclusions

In conclusion the author is convinced that the MIDO package **DESolve0.m** will be a useful extension of the built-in procedure **DSolve** in *Mathematica*. The procedure **initialValueProblem** turns out to be a valuable procedure for treating initial value problems for the Laplace and wave equation. Similarly, the **BIVProblem** copes with 8 different types of boundary conditions and various types of the initial value problems of the 1d/2d-heat/diffusion equation.

It will be beyond the scope of this article to demonstrate the extension of MIDO to *systems of linear, homogeneous PDEs*. Calculations of systems of 2 respective 3 coupled PDEs had been performed. Moreover, this method can also be used for systems of coupled heat/diffusion equations. In addition, initial values are applied to the solutions obtained for systems of coupled PDEs. In the case of coupled heat/diffusion equations boundary conditions together with initial values are applied to the decoupled solutions $w_i(t, x_i)$ (with $x_i = x \,||\, y \,||\, z$) which are then transformed back into the corresponding solutions $u_i(t, x, y, z)$ of the coupled system.

Appendix

(1) Gaussian and Quaternion Factorization

(2) Distinction between *function* Abs[x] and *distribution* abs[x]

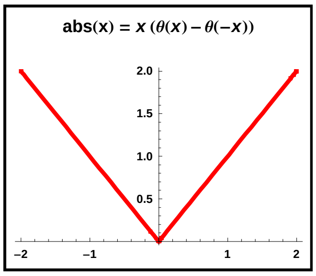
According to a private communication with Michael Trott / WRI there is a subtlety as regards to derivatives of the built-in function **Abs**; **Abs'[x]** is not automatically evaluated to **Sign[x]** because generally $x \in \mathbb{C}$ is assumed. Only for $x \in \mathbb{R}$ the result will be **Abs'[x]= Sign[x]**. This is easily demonstrated :

```
{ Abs'[x]// FullSimplify[#, x ∈ R ]&,
  Abs'[x]// FullSimplify[#, x ∈ C ]& }
```

```
{Sign[x], Abs′[x]}
```

However, if one wants to express the derivatives of the built-in function **Abs** in terms of **DiracDelta** functions then one has to resort to *distributions*. The *Mathematica* function **Abs[x]** has to be redefined in terms of a *distribution* **abs[x]** such that :

```
abs[x_]:= x (HeavisideTheta[x] - HeavisideTheta[-x])
Framed[Plot[abs[x],{x,-2,2},
            PlotLabel →Style[StringJoin["abs(x) = ",ToString[abs[x],tF],"\n"],9],
            LabelStyle →Directive[Bold,Tiny,FontFamily →"Helvetica"],ImageSize →150]]
```



The first few derivatives of the distribution **abs[x]** turn out to be :

```
FullSimplify @ Table[∂{x,n}abs[x],{n,0,5}]//tF
```

$\{x\,(\theta(x)-\theta(-x)),\ \theta(x)-\theta(-x),\ 2\,\delta(x),\ 2\,\delta'(x),\ 2\,\delta''(x),\ 2\,\delta^{(3)}(x)\}$

As regards to the representation of the unit step function note the distinction that procedure **UnitStep** is a *function* whereas **HeavisideTheta** is a *distribution* . Therefore, w.r.t. **x=0** one obtains for **UnitStep[0]=1** , whereas the value for **HeavisideTheta[0]** is not defined.

```
? UnitStep HeavisideTheta
```

Acknowledgments

The author would like to thank Vladimir Gerdt from JINR, Dubna/Russia for his suggestions and Leonid Shifrin, WRI consultant in St. Petersburg/Russia, for his patient support and competent solutions regarding sophisticated problems in *Mathematica*. Special thanks to Jason Harris, Michael Trott and Oliver Rübenkönig, all at WRI, for their valuable contributions and suggestions for notational improvements during the development of the implementation. Finally, the constructive help of David Park concerning the structure of *Mathematica* packages is gratefully acknowledged.

References

P. K. Kythe, **P. Puri** & **M. R. Schäferkotter**, "Partial Differential Equations and Boundary Value Problems with Mathematica", 2nd Ed. , Chapman & Hall / CRC, 2002, Chapter 3.2-3.4, pp. 75-81.

R. Kragler, "Method of Inverse Differential Operators for the Solution of PDEs," in *Computer Algebra Systems in Teaching and Research, 6th International. Workshop,* vol. Differential Equations, Dynamical Systems and Celestial Mechanics (CASTR 2011), Siedlce (L. Gadomski et al. eds.), Siedlce : Wydawnictwo Collegium Mazovia, 2011 pp. 79-95.

R. Kragler, "Method of Inverse Differential Operators applied to certain classes of non-homogeneous PDEs and ODEs", in Proceedings of "The Third International Conference : Mathematical Modelling and Differential Equations", Brest State University, Brest/Belarus (Sept. 2012), Publishing Center BSU, Minsk, 2012 pp. 290-307, ISBN 978-985-553-054-2

Martha L Abell & James P. Braselton, "Differential Equations with *Mathematica*", 2nd Ed. , Academic Press , 1997, Chapter 11.2, pp. 723-737.

Selwyn Hollis, " A *Mathematica* Companion for Differential Equations" 2nd Ed., Prentice Hall/Pearson Education , 2003, Chapter 11.1, pp. 227-233.