Refined constitutive shell equations with *MathTensor*

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Abstract

The paper[1] presents the results of research on refinement of constitutive shell equations with the *MathTensor* package of the *Mathematica* system. The obtained results satisfy the last equation of equilibrium and better describe the shell behaviour.

1 Introduction

Theory of shells is a part of theory of elasticity which deals with description and analysis of thin curved elastic bodies called shells. It is based on differential geometry and uses as a tool tensor analysis. The main aim of the theory is to reduce the three dimensional task of the theory of elasticity in the curved space to two dimensional one. It makes possible its practical application to civil and mechanical engineering.

The shell description is based on so called reference surface. All points of the shell are referred to this surface before and after deformation. Usually this reference is done to mid-surface which lies in the middle between bounding surfaces.

The constitutive relations between stress tensor τ^{ij} and strain tensor $\overset{\star}{\gamma}_{ij}$ in the 3D body are reduced to the relations between two dimensional strain tensors of

[1] The document is available in PostScript (PS) and Portable Document Format (PDF) versions.

the reference surface γ_{ij}, ρ_{ij}, ϑ_{ij} and internal forces tensors: axial forces N^{ij} and moments M^{ij}.

The asymmetrical tensors of axial forces and moments to be computed from the following integrals [1], respectively

$$N^{ij} = \int\limits_{-h}^{h} \sqrt{\frac{\boldsymbol{g}}{\boldsymbol{a}}} \, \left(\delta_p^j - z\, b_p^j\right) \, \tau^{ip} \, \mathrm{d}z, \tag{1}$$

$$M^{ij} = \int\limits_{-h}^{h} \sqrt{\frac{\boldsymbol{g}}{\boldsymbol{a}}} \, \left(\delta_p^j - z\, b_p^j\right) \, \tau^{ip} \, z \, \mathrm{d}z. \tag{2}$$

These integrals become very complicated after substituting into them formulas onto $\boldsymbol{g}$ and τ^{ij}. They take the following forms

$$\begin{aligned} N^{ij} = \int\limits_{-h}^{h} & \left(\delta_r^j - z\, b_r^j\right) \frac{E\left(\gamma_{pq} - 2\, z\, \rho_{pq} + z^2\, \vartheta_{pq}\right)}{\left(1-\nu^2\right)\left(1 - 2\, H\, z + K\, z^2\right)^3} \cdot \\ & \cdot \Big\{ \left(1 - 4\, H\, z + 4\, H^2\, z^2 - K\, z^2\right)^2 \cdot \\ & \cdot \left[\nu\, a^{pq}\, a^{ir} + (1-\nu)\, a^{pi}\, a^{qr}\right] + \\ & - 2\, z\, (1 - H\, z) \left(1 - 4\, H\, z + 4\, H^2\, z^2 - K\, z^2\right) \cdot \\ & \cdot \left[\nu\, \left(b^{pq}\, a^{ir} + a^{pq}\, b^{ir}\right) + (1-\nu) \left(b^{pi}\, a^{qr} + a^{pi}\, b^{qr}\right)\right] + \\ & + 4\, z^2\, (1 - H\, z)^2 \left[\nu\, b^{pq}\, b^{ir} + (1-\nu)\, b^{pi}\, b^{qr}\right] \Big\} \, \mathrm{d}z, \end{aligned} \tag{3}$$

$$\begin{aligned} M^{ij} = \int\limits_{-h}^{h} & \left(\delta_r^j - z\, b_r^j\right) z \frac{E\left(\gamma_{pq} - 2\, z\, \rho_{pq} + z^2\, \vartheta_{pq}\right)}{\left(1-\nu^2\right)\left(1 - 2\, H\, z + K\, z^2\right)^3} \cdot \\ & \cdot \Big\{ \left(1 - 4\, H\, z + 4\, H^2\, z^2 - K\, z^2\right)^2 \cdot \\ & \cdot \left[\nu\, a^{pq}\, a^{ir} + (1-\nu)\, a^{pi}\, a^{qr}\right] + \\ & - 2\, z\, (1 - H\, z) \left(1 - 4\, H\, z + 4\, H^2\, z^2 - K\, z^2\right) \cdot \\ & \cdot \left[\nu\, \left(b^{pq}\, a^{ir} + a^{pq}\, b^{ir}\right) + (1-\nu) \left(b^{pi}\, a^{qr} + a^{pi}\, b^{qr}\right)\right] + \\ & + 4\, z^2\, (1 - H\, z)^2 \left[\nu\, b^{pq}\, b^{ir} + (1-\nu)\, b^{pi}\, b^{qr}\right] \Big\} \, \mathrm{d}z. \end{aligned} \tag{4}$$

Therefore in shell theories several simplification were done in (1) and (2) evaluation. The most of theories, including geometrically non-linear, apply the formulas

$$N^{ij} \approx \bar{N}^{ij} = \frac{2\,E\,h}{1-\nu^2}\left[\nu\,a^{ij}\,\gamma^p_p + (1-\nu)\,\gamma^{ij}\right], \tag{5}$$

$$M^{ij} \approx \hat{M}^{ij} = \frac{4\,E\,h^3}{3\,(1-\nu^2)}\left[\nu\,a^{ij}\,\rho^p_p + (1-\nu)\,\rho^{ij}\right]. \tag{6}$$

The symmetrical tensors $\bar{N}^{ij}$ and $\hat{M}^{ij}$ are simple but they do not satisfy the last equation of equilibrium

$$\epsilon_{pq}\,(N^{pq} - b^q_r\,M^{rp}) = 0. \tag{7}$$

This problem was not possible to be solved properly without computer assistance due to algebraic difficulties. The paper presents the refined evaluation of the integrals (1) and (2). It was done with an assistance of *Mathematica* and its external *MathTensor* package. *MathTensor* is a product of MathSolutions, Inc. It was written by Steven M. Christensen and Leonard Parker [2].

The next section presents brief theoretical explanation with main steps of this evaluation. The full computational process can be studied from the attached notebook `Evaluation of refined tensors.nb`. It can be read with Publicon 0.9 BETA, for details please refer to `http://www.publicon.com/`

All *Mathematica* inputs will be denoted with `blue typewriter text` for emphasis.

2 Evaluation with *MathTensor*

The *Mathematica* session should start with loading of the *MathTensor* package. It is done with a command[2].

```
<<MathTens.m
```

To obtain better output we can format standard output of Kronecker delta with:

```
Format[Kdelta[a__]]:=PrettyForm[δ, a]
```

[2] On Dos/Windows system

2.1 Geometrical description

To describe the mid-surface we define two symmetrical tensors a_{ij} and b_{ij}, their coefficients are called coefficients of the differential forms. Tensor a_{ij} is a metric tensor on the mid-surface so it can be absorbed in tensor operations.

```
DefineTensor[a,"a",{{2,1},1}]

DefineTensor[b,"b",{{2,1},1}]
```

We introduce two scalars useful in the shell analysis called curvatures - Gaussian and average one, which are defined by:

$$K = \frac{\boldsymbol{b}}{\boldsymbol{a}} \tag{8}$$

and

$$H = \frac{1}{2}\, a^{ij}\, b_{ij} \tag{9}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are determinants of the tensors a_{ij} and b_{ij} respectively.

Contravariant metric tensor in the 3D space is defined by:

$$g^{ij} = \frac{a^{ij}\,(1 - K\,z^2) - 2\,(2\,H\,a^{ij} - b^{ij})\,(1 - H\,z)\,z}{(1 - 2\,H\,z + K\,z^2)^2} \tag{10}$$

```
Metricg[ui_, uj_] :=
  (a[ui, uj]*(1 - K*z^2)
    - 2*(2*H*a[ui, uj] - b[ui, uj])*(1 - H*z)*z)/
   (1 - 2*H*z + K*z^2)^2
```

2.2 Strain tensor

Each of the strain tensors presented below is a measure of change of the a_{ij}, b_{ij} and c_{ij} mid-surface differential forms during deformation. These tensors are symmetrical.

```
DefineTensor[gamma,"γ",{{2,1},1}]

DefineTensor[rho,"ρ",{{2,1},1}]
```

```
DefineTensor[theta,"ϑ",{{2,1},1}]
```

Strain tensor in the 3D space which describes the change of the metric tensor

```
DefineTensor[gammastar,"γs",{{2,1},1}]
```

is defined by

$$\overset{\star}{\gamma}_{ij} = \gamma_{ij} - 2\,z\,\rho_{ij} + z^2\,\vartheta_{ij} \tag{11}$$

```
gammastar[li_, lj_] :=
   gamma[li, lj] - 2*z*rho[li,lj]+z^2*theta[li, lj]
```

2.3 Stress tensor

Stress tensor in a 3D curved space is symmetrical, too.

```
DefineTensor[tau,"τ",{{2,1},1}]
```

For linear-elastic material constitutive stress-strain relation in a curved space has the following definition:

$$\tau^{ij} = \left[\overset{\star}{\lambda}\,g^{ij}\,g^{pq} + 2\,\mu\,g^{ip}\,g^{jq}\right]\overset{\star}{\gamma}_{pq} \tag{12}$$

where

$$\overset{\star}{\lambda} = \frac{\nu\,E}{1-\nu^2}$$

and

$$\mu = \frac{E}{2(1+\nu)}$$

```
tau[ui_, uj_] :=
   (λ*Metricg[ui, uj]*Metricg[u1, u2] +
    2*μ*Metricg[ui,u1]*Metricg[uj,u2])*
      gammastar[l1,l2]
```

2.4 Tensors of internal forces

We introduce the function of integration to speed up computation. This function could be mapped over the expanded expression which helps *Mathematica* in integrals evaluation.

```
integ[x_] := Integrate[x, {z, -h, h}]
```

Tensors of axial forces N^{ij} and moments M^{ij} according to the definitions (1) and (2) are not symmetrical.

```
DefineTensor[n,"n",{{1,2},1}]

DefineTensor[m,"m",{{1,2},1}]
```

They can be evaluated with an arbitrary precision of the power expansion of the shell thickness $2\,h$ by the following *Mathematica/MathTensor* definitions:

```
n[ui_,uj_][k_]:=
Simplify/@Tsimplify[
 AbsorbKdelta[
  integ/@Expand[
   Normal[Series[
    (1-2*H*z+K*z^2)*
       (Kdelta[l3,uj]-z*b[l3,uj])*tau[ui,u3],
   {z,0,k}]]
  ]
 ]
]


m[ui_,uj_][k_]:= Simplify/@Tsimplify[
 AbsorbKdelta[
  integ/@Expand[
   Normal[Series[
    (1-2*H*z+K*z^2)*
       (Kdelta[l3,uj]-z*b[l3,uj])*tau[ui,u3]*z,
   {z,0,k}]]
  ]
 ]
]
```

In these definitions the integrated functions are expanded into power series.

```
Normal[Series[...]]
```

Then, expanded expression is integrated term by term.

```
integ/@Expand[...]
```

Next, Kronecker delta is absorbed.

```
AbsorbKdelta[...]
```

In the end the result is simplified.

```
Simplify/@Tsimplify[...]
```

The parameter `k` defines the order of z variable power expansion. The result has $k+1$ precision of the shell thickness expansion. Moreover, if the reference surface is a mid-one then terms with even terms of the shell thickness power expansion (h^2, $h^4, \ldots$) disappear.

3 Results

The results can be received with an arbitrary precision. Precision to the third power of the thickness is enough for engineering purposes. It is obtained by the following inputs:

```
n[ui,uj][2]
```

```
m[ui,uj][2]
```

If these expressions are collected with

```
Collect[expr,{gamma[l1,l2],rho[l1,l2],theta[l1,l2]}]
```

we receive the following results:

–the relation between axial forces tensor and strain tensors

$$
\begin{aligned}
N^{ij} = & \left[\frac{2\,E\,h\,\left(3-5\,K\,h^2\right)\,a^{pj}\,a^{qi}}{3\,\left(1+\nu\right)} + \frac{2\,E\,h\,\left(3-5\,K\,h^2\right)\,\nu\,a^{pq}\,a^{ij}}{3\,\left(1-\nu^2\right)} + \right.\\
& + \frac{4\,E\,H\,h^3\,a^{pr}\,a^{qi}\,b^j_r}{3\,\left(1+\nu\right)} + \frac{4\,E\,H\,h^3\,\nu\,a^{pq}\,a^{ri}\,b^j_r}{3\,\left(1-\nu^2\right)} + \frac{4\,E\,H\,h^3\,\nu\,a^{ij}\,b^{pq}}{3-3\,\nu^2} + \\
& - \frac{4\,E\,h^3\,\nu\,a^{ri}\,b^j_r\,b^{pq}}{3\,\left(1-\nu^2\right)} - \frac{4\,E\,h^3\,a^{qi}\,b^j_r\,b^{pr}}{3\,\left(1+\nu\right)} + \frac{4\,E\,H\,h^3\,a^{qi}\,b^{pj}}{3\,\left(1+\nu\right)} + \\
& + \frac{4\,E\,H\,h^3\,a^{pj}\,b^{qi}}{3\,\left(1+\nu\right)} - \frac{4\,E\,h^3\,a^{pr}\,b^j_r\,b^{qi}}{3\,\left(1+\nu\right)} + \frac{8\,E\,h^3\,b^{pj}\,b^{qi}}{3\,\left(1+\nu\right)} + \\
& \left. - \frac{4\,E\,h^3\,\nu\,a^{pq}\,b^j_r\,b^{ri}}{3\,\left(1-\nu^2\right)} + \frac{4\,E\,H\,h^3\,\nu\,a^{pq}\,b^{ij}}{3\,\left(1-\nu^2\right)} + \frac{8\,E\,h^3\,\nu\,b^{pq}\,b^{ij}}{3\,\left(1-\nu^2\right)}\right]\gamma_{pq} + \\
& + \left[\frac{8\,E\,H\,h^3\,a^{pj}\,a^{qi}}{3\,\left(1+\nu\right)} + \frac{8\,E\,H\,h^3\,\nu\,a^{pq}\,a^{ij}}{3\,\left(1-\nu^2\right)} + \frac{4\,E\,h^3\,a^{pr}\,a^{qi}\,b^j_r}{3\,\left(1+\nu\right)} + \right.\\
& + \frac{4\,E\,h^3\,\nu\,a^{pq}\,a^{ri}\,b^j_r}{3\,\left(1-\nu^2\right)} - \frac{8\,E\,h^3\,\nu\,a^{ij}\,b^{pq}}{3\,\left(1-\nu^2\right)} - \frac{8\,E\,h^3\,a^{qi}\,b^{pj}}{3\,\left(1+\nu\right)} + \\
& \left. - \frac{8\,E\,h^3\,a^{pj}\,b^{qi}}{3\,\left(1+\nu\right)} - \frac{8\,E\,h^3\,\nu\,a^{pq}\,b^{ij}}{3\,\left(1-\nu^2\right)}\right]\rho_{pq} + \\
& + \left[\frac{2\,E\,h^3\,a^{pj}\,a^{qi}}{3\,\left(1+\nu\right)} + \frac{2\,E\,h^3\,\nu\,a^{pq}\,a^{ij}}{3\,\left(1-\nu^2\right)}\right]\vartheta_{pq} + O\left(h^5\right)
\end{aligned}
\tag{13}
$$

–the constitutive relation for moment tensor

$$
\begin{aligned}
M^{ij} = & \left[\frac{4\,E\,h^3\,\nu\,a^{ij}\,b^{pq}}{3\,\left(1-\nu^2\right)} + \frac{4\,E\,h^3\,a^{qi}\,b^{pj}}{3\,\left(1+\nu\right)} + \frac{4\,E\,h^3\,a^{pj}\,b^{qi}}{3\,\left(1+\nu\right)} + \right.\\
& + \frac{4\,E\,h^3\,\nu\,a^{pq}\,b^{ij}}{3\,\left(1-\nu^2\right)} - \frac{4\,E\,H\,h^3\,a^{pj}\,a^{qi}}{3\,\left(1+\nu\right)} - \frac{4\,E\,H\,h^3\,\nu\,a^{pq}\,a^{ij}}{3\,\left(1-\nu^2\right)} + \\
& \left. - \frac{2\,E\,h^3\,a^{pr}\,a^{qi}\,b^j_r}{3\,\left(1+\nu\right)} - \frac{2\,E\,h^3\,\nu\,a^{pq}\,a_{ri}\,b^j_r}{3\,\left(1-\nu^2\right)}\right]\gamma_{pq} + \\
& - \left[\frac{4\,E\,h^3\,a^{pj}\,a^{qi}}{3\,\left(1+\nu\right)} + \frac{4\,E\,h^3\,\nu\,a^{pq}\,a^{ij}}{3\,\left(1-\nu^2\right)}\right]\rho_{pq} + O\left(h^5\right).
\end{aligned}
\tag{14}
$$

The refined constitutive relations (13) and (14) satisfy the last equation of equilibrium (7) which can be verified with

```
Simplify/@Tsimplify[
 Expand[
  EpsDown[li,lj]*(n[ui,uj] - m[uk,uj]*b[lk,ui])
 ]
]=0
```

The results can be used directly in further calculations but they can be simplified. Application of `Absorb[]`, `Canonicalize[]`, `Tsimplify[]` and some other algebraic manipulations lead to the following formulas:

–axial forces tensor

$$
\begin{aligned}
N^{ij} = \frac{2\,E\,h}{3\,(1-\nu^2)} \Big\{ & \left(3 - 5\,K\,h^2\right) \left[\nu\, a^{ij}\, \gamma^p_p + (1-\nu)\,\gamma^{ij}\right] + \\
& + 2\left[\nu\, b_{pq}\, b^{ij} + (1-\nu)\, b^i_p\, b^j_q\right] \gamma^{ij}\, h^2 + \\
& + 4\,H \left[\nu\, b^{ij}\, \gamma^p_p + (1-\nu)\, b^j_p \gamma^{pi}\right] h^2 + \\
& + 2\,H \left[\nu\, a^{ij}\, b_{pq}\, \gamma^{pq} + (1-\nu)\, b^i_p \gamma^{pj}\right] h^2 + \\
& - 2\, b^{pj} \left[\nu\, b^i_p\, \gamma^q_q + (1-\nu)\, b_{pq} \gamma^{qi}\right] h^2 + \\
& - 4\left[\nu\, a^{ij}\, b_{pq}\, \rho^{pq} + (1-\nu)\, b^i_p\, \rho^{pj}\right] h^2 + \\
& + \left(2\,H\,\delta^j_q - b^j_q\right) \left[\nu\, a^{qi}\, \rho^p_p + (1-\nu)\,\rho^{qi}\right] h^2 + \\
& - 2\left[\nu\, a^{ij}\, \vartheta^p_p + (1-\nu)\,\vartheta^{ij}\right] h^2 \Big\} + O\left(h^5\right)
\end{aligned}
\tag{15}
$$

–moment tensor

$$
\begin{aligned}
M^{ij} = \frac{2\,E\,h^3}{3\,(1-\nu^2)} \Big\{ & 2\left[\nu\, a^{ij}\, b_{pq}\, \gamma^{pq} + (1-\nu)\, b^i_p\, \gamma^{pj}\right] + \\
& - \left(2\,H\,\delta^j_q - b^j_q\right) \left[\nu\, a^{qi}\, \gamma^p_p + (1-\nu)\,\gamma^{qi}\right] + \\
& - 2\left[\nu\, a^{ij}\, \rho^p_p + (1-\nu)\,\rho^{ij}\right] \Big\} + O\left(h^5\right).
\end{aligned}
\tag{16}
$$

The obtained formulas can be applied both in linear and geometrically nonlinear theories of shells because they are relations between internal forces and strain tensors. The approach can be easily expanded to multilayered shells. It is also possible to obtain the formulas with higher precision. The input:

```
m[ui,uj][4]
```

yields the result for moment tensor with a precision to h^5

$$
\begin{aligned}
M^{ij} = {} & \frac{-4E\,h^3\,H\,(5+6h^2\,K)\,a^{pj}\,a^{qi}\,\gamma_{pq}}{15\,(1+\nu)} - \frac{4E\,h^3 H\,(5+6h^2\,K)\,\nu\,a^{pq}\,a^{ij}\,\gamma_{pq}}{15\,(1-\nu^2)} + \\
& - \frac{2\,E\,h^3\,(1-3\,h^2\,K)\,a^{pr}\,a^{qi}\,b^j_r\,\gamma_{pq}}{3\,(1+\nu)} - \frac{2\,E\,h^3\,(1-3\,h^2\,K)\,\nu\,a^{pq}\,a^{ri}\,b^j_r\,\gamma_{pq}}{3\,(1-\nu^2)} + \\
& + \frac{4\,E\,h^3\,\left(5+6\,h^2\,\left(H^2-2\,K\right)\right)\,\nu\,a^{ij}\,b^{pq}\,\gamma_{pq}}{15\,(1-\nu^2)} - \frac{4\,E\,h^5\,H\,\nu\,a^{ri}\,b^j_r\,b^{pq}\,\gamma_{pq}}{5\,(1-\nu^2)} + \\
& - \frac{4\,E\,h^5\,H\,a^{qi}\,b^j_r\,b^{pr}\,\gamma_{pq}}{5\,(1+\nu)} + \frac{4\,E\,h^3\,\left(5+6\,h^2\,\left(H^2-2\,K\right)\right)\,a^{qi}\,b^{pj}\,\gamma_{pq}}{15\,(1+\nu)} + \\
& + \frac{4\,E\,h^3\,\left(5+6\,h^2\,\left(H^2-2\,K\right)\right)\,a^{pj}\,b^{qi}\,\gamma_{pq}}{15\,(1+\nu)} - \frac{4\,E\,h^5\,H\,a^{pr}\,b^j_r\,b^{qi}\,\gamma_{pq}}{5\,(1+\nu)} + \\
& - \frac{8\,E\,h^5\,b^j_r\,b^{pr}\,b^{qi}\,\gamma_{pq}}{5\,(1+\nu)} + \frac{32\,E\,h^5\,H\,b^{pj}\,b^{qi}\,\gamma_{pq}}{5\,(1+\nu)} - \frac{4\,E\,h^5\,H\,\nu\,a^{pq} b^j_r\,b^{ri}\,\gamma_{pq}}{5\,(1-\nu^2)} + \\
& - \frac{8\,E\,h^5\,\nu\,b^j_r\,b^{pq} b^{ri}\,\gamma_{pq}}{5\,(1+\nu^2)} + \frac{4\,E\,h^3\,\left(5+6\,h^2\,\left(H^2-2\,K\right)\right)\,\nu\,a^{pq} b^{ij}\,\gamma_{pq}}{15\,(1-\nu^2)} + \\
& + \frac{32\,E\,h^5\,H\,\nu\,b^{pq} b^{ij}\,\gamma_{pq}}{5\,(1-\nu^2)} - \frac{4\,E\,h^5\,H\,a^{pj}\,a^{qi}\,\vartheta_{pq}}{5\,(1+\nu)} - \frac{4\,E\,h^5\,H\,\nu\,a^{pq} a^{ij}\,\vartheta_{pq}}{5\,(1-\nu^2)} + \\
& - \frac{2\,E\,h^5\,a^{pr}\,a^{qi}\,b^j_r\,\vartheta_{pq}}{5\,(1+\nu)} - \frac{2\,E\,h^5\,\nu\,a^{pq} a^{ri}\,b^j_r\,\vartheta_{pq}}{5\,(1-\nu^2)} + \frac{4\,E\,h^5\,\nu\,a^{ij}\,b^{pq}\vartheta_{pq}}{5\,(1-\nu^2)} + \\
& + \frac{4\,E\,h^5\,a^{qi}\,b^{pj}\,\vartheta_{pq}}{5\,(1+\nu)} + \frac{4\,E\,h^5\,a^{pj}\,b^{qi}\,\vartheta_{pq}}{5\,(1+\nu)} + \frac{4\,E\,h^5\,\nu\,a^{pq} b^{ij}\,\vartheta_{pq}}{5\,(1-\nu^2)} + \\
& - \frac{4\,E\,h^3\,\left(1-3\,h^2\,K\right)\,a^{pj}\,a^{qi}\,\rho_{pq}}{3\,(1+\nu)} - \frac{4\,E\,h^3\,\left(1-3\,h^2\,K\right)\,\nu\,a^{pq} a^{ij}\,\rho_{pq}}{3\,(1-\nu^2)} + \\
& - \frac{8\,E\,h^5 H\,a^{pr}\,a^{qi}\,b^j_r\,\rho_{pq}}{5\,(1+\nu)} - \frac{8\,E\,h^5 H\,\nu\,a^{pq} a^{ri}\,b^j_r\,\rho_{pq}}{5\,(1-\nu^2)} - \frac{8\,E\,h^5 H\,\nu\,a^{ij} b^{pq}\rho_{pq}}{5\,(1-\nu^2)} + \\
& + \frac{8\,E\,h^5\,\nu\,a^{ri}\,b^j_r\,b^{pq}\rho_{pq}}{5\,(1-\nu^2)} + \frac{8\,E\,h^5\,a^{qi}\,b^j_r\,b^{pr}\,\rho_{pq}}{5\,(1+\nu)} - \frac{8\,E\,h^5\,H\,a^{qi}\,b^{pj}\,\rho_{pq}}{5\,(1+\nu)} + \\
& - \frac{8\,E\,h^5\,H\,a^{pj}\,b^{qi}\,\rho_{pq}}{5\,(1+\nu)} + \frac{8\,E\,h^5\,a^{pr}\,b^j_r\,b^{qi}\,\rho_{pq}}{5\,(1+\nu)} - \frac{16\,E\,h^5\,b^{pj}\,b^{qi}\,\rho_{pq}}{5\,(1+\nu)} + \\
& + \frac{8\,E\,h^5\,\nu\,a^{pq} b^j_r\,b^{ri}\,\rho_{pq}}{5\,(1-\nu^2)} - \frac{8\,E\,h^5\,H\,\nu\,a^{pq} b^{ij}\,\rho_{pq}}{5\,(1-\nu^2)} - \frac{16\,E\,h^5\,\nu\,b^{pq} b^{ij}\,\rho_{pq}}{5\,(1-\nu^2)} + \\
& + O\left(h^7\right)
\end{aligned} \tag{17}
$$

4 Kinematic relations

The received refined formulas seem to be more complex than simplified ones but after substituting linear kinematic relations into them they become simpler.

For example for spherical shell the kinematic relations substitution results in the following formulas for M^{ij} moment tensor components:

– simplified

$$\begin{aligned}\hat{M}^{11} = &+\frac{4\,E\,h^3\,d^{\neg}_{1,1}}{3\,(1-\nu^2)} + \frac{4\,E\,h^3\,\nu\,\sec\left(\frac{x}{s_o}\right)\,d^{\neg}_{2,2}}{3\,(1-\nu^2)\,s_o} + \\ &-\frac{4\,E\,h^3\,\nu\,\tan\left(\frac{x}{s_o}\right)\,d^{\neg}_{1}}{3\,(1-\nu^2)\,s_o} - \frac{4\,E\,h^3\,w^{\neg}_{1,1}}{3\,(1-\nu^2)\,s_o} + \\ &-\frac{4\,E\,h^3\,\nu\,\sec\left(\frac{x}{s_o}\right)\,w^{\neg}_{2,2}}{3\,(1-\nu^2)\,{s_o}^2} + \frac{4\,E\,h^3\,\nu\,\tan\left(\frac{x}{s_o}\right)\,w^{\neg}_{1}}{3\,(1-\nu^2)\,{s_o}^2} + \\ &+\frac{2\,E\,h^3\,w^{\neg}_{3}}{3\,(1-\nu)\,{s_o}^2} + \frac{4\,E\,h^3\,\varepsilon_t}{3\,(1-\nu)\,s_o} + \frac{2\,E\,h^3\,\kappa_t}{3\,(1-\nu)},\end{aligned} \tag{18}$$

$$\begin{aligned}\hat{M}^{12} = \hat{M}^{21} = &\frac{2\,E\,h^3\sec^2\left(\frac{x}{s_o}\right)\,d^{\neg}_{1,2}}{3\,(1+\nu)\,{s_o}^2} + \frac{2\,E\,h^3\,\sec\left(\frac{x}{s_o}\right)\,d^{\neg}_{2,1}}{3\,(1+\nu)\,s_o} + \\ &+\frac{2\,E\,h^3\,\sec\left(\frac{x}{s_o}\right)\,\tan\left(\frac{x}{s_o}\right)\,d^{\neg}_{2}}{3\,(1+\nu)\,{s_o}^2} - \frac{2\,E\,h^3\sec^2\left(\frac{x}{s_o}\right)\,w^{\neg}_{1,2}}{3\,(1+\nu)\,{s_o}^3} + \\ &-\frac{2\,E\,h^3\,\sec\left(\frac{x}{s_o}\right)\,w^{\neg}_{2,1}}{3\,(1+\nu)\,{s_o}^2} - \frac{2\,E\,h^3\,\sec\left(\frac{x}{s_o}\right)\,\tan\left(\frac{x}{s_o}\right)\,{w_2}^{\neg}}{3\,(1+\nu)\,{s_o}^3},\end{aligned} \tag{19}$$

$$\begin{aligned}\hat{M}^{22} = &+\frac{4\,E\,h^3\,\nu\sec^2\left(\frac{x}{s_o}\right)d^{\neg}_{1,1}}{3\,(1-\nu^2)\,{s_o}^2} + \frac{4\,E\,h^3\sec^3\left(\frac{x}{s_o}\right)d^{\neg}_{2,2}}{3\,(1-\nu^2)\,{s_o}^3} + \\ &-\frac{4\,E\,h^3\sec^2\left(\frac{x}{s_o}\right)\,\tan\left(\frac{x}{s_o}\right)\,d^{\neg}_{1}}{3\,(1-\nu^2)\,{s_o}^3} - \frac{4\,E\,h^3\,\nu\sec^2\left(\frac{x}{s_o}\right)w^{\neg}_{1,1}}{3\,(1-\nu^2)\,{s_o}^3} + \\ &-\frac{4\,E\,h^3\sec^3\left(\frac{x}{s_o}\right)w^{\neg}_{2,2}}{3\,(1-\nu^2)\,{s_o}^4} + \frac{4\,E\,h^3\sec^2\left(\frac{x}{s_o}\right)\,\tan\left(\frac{x}{s_o}\right)\,w^{\neg}_{1}}{3\,(1-\nu^2)\,{s_o}^4} + \\ &+\frac{2\,E\,h^3\sec^2\left(\frac{x}{s_o}\right)w^{\neg}_{3}}{3\,(1-\nu)\,{s_o}^4} + \frac{4\,E\,h^3\sec^2\left(\frac{x}{s_o}\right)\varepsilon_t}{3\,(1-\nu)\,{s_o}^3} + \frac{2\,E\,h^3\sec^2\left(\frac{x}{s_o}\right)\kappa_t}{3\,(1-\nu)\,{s_o}^2},\end{aligned} \tag{20}$$

– refined

$$M^{11} = \frac{4\,E\,h^3\,d^{\neg}_{1,1}}{3\,(1-\nu^2)} + \frac{4\,E\,h^3\,\nu\,\sec\left(\frac{x}{s_o}\right)\,d^{\neg}_{2,2}}{3\,(1-\nu^2)\,s_o} + \\ - \frac{4\,E\,h^3\,\nu\,\tan\left(\frac{x}{s_o}\right)\,d^{\neg}_{1}}{3\,(1-\nu^2)\,s_o} + \frac{2\,E\,h^3\,\varepsilon_t}{3\,s_o\,(1-\nu)} + \frac{2\,E\,h^3\,\kappa_t}{3\,(1-\nu)}, \tag{21}$$

$$M^{12} = M^{21} = \frac{2\,E\,h^3\,\sec^2\left(\frac{x}{s_o}\right)\,d^{\neg}_{1,2}}{3\,(1+\nu)\,{s_o}^2} + \frac{2\,E\,h^3\,\sec\left(\frac{x}{s_o}\right)\,d^{\neg}_{2,1}}{3\,(1+\nu)\,s_o} + \\ + \frac{2\,E\,h^3\,\sec\left(\frac{x}{s_o}\right)\,\tan\left(\frac{x}{s_o}\right)\,d^{\neg}_{2}}{3\,(1+\nu)\,{s_o}^2}, \tag{22}$$

$$M^{22} = \frac{4\,E\,h^3\,\nu\,\sec^2\left(\frac{x}{s_o}\right)\,d^{\neg}_{1,1}}{3\,(1-\nu^2)\,{s_o}^2} + \frac{4\,E\,h^3\,\sec^3\left(\frac{x}{s_o}\right)\,d^{\neg}_{2,2}}{3\,(1-\nu^2)\,{s_o}^3} + \\ - \frac{4\,E\,h^3\,\sec^2(\frac{x}{s_o})\,\tan\left(\frac{x}{s_o}\right)\,d^{\neg}_{1}}{3\,(1-\nu^2)\,{s_o}^3} + \frac{2\,E\,h^3\,\sec^2\left(\frac{x}{s_o}\right)\,\varepsilon_t}{3\,(1-\nu)\,{s_o}^3} + \\ + \frac{2\,E\,h^3\,\sec^2\left(\frac{x}{s_o}\right)\,\kappa_t}{3\,(1-\nu)\,{s_o}^2}. \tag{23}$$

The refined formulas are shorter and depend only on $d^{\neg}_i$ vector of rotation and thermal influences κ_t, ε_t. The simplified formulas contain terms with a displacement vector $w^{\neg}_i$, additionally. Coefficients by ε_t in both types of expressions are different. Thus, in certain circumstances application of simplified constitutive relations may result in significant numerical error.

5 Numerical example

The evaluated constitutive equations do not only satisfy the last equation of equilibrium but they better describe the shell behaviour. It will be illustrated with an example of large deformation of a plate, see Fig. 1. The Young modulus for the material $E = 3000\;kPa$, Poisson ratio $\nu = \frac{1}{6}$ and the plate thickness $2\,h = 0.1\,m$.

Let us consider the internal forces associated with this deformation. There were shown in the Figs 2 and 3 that the results for the same deformation are significantly different for simplified and refined approaches.

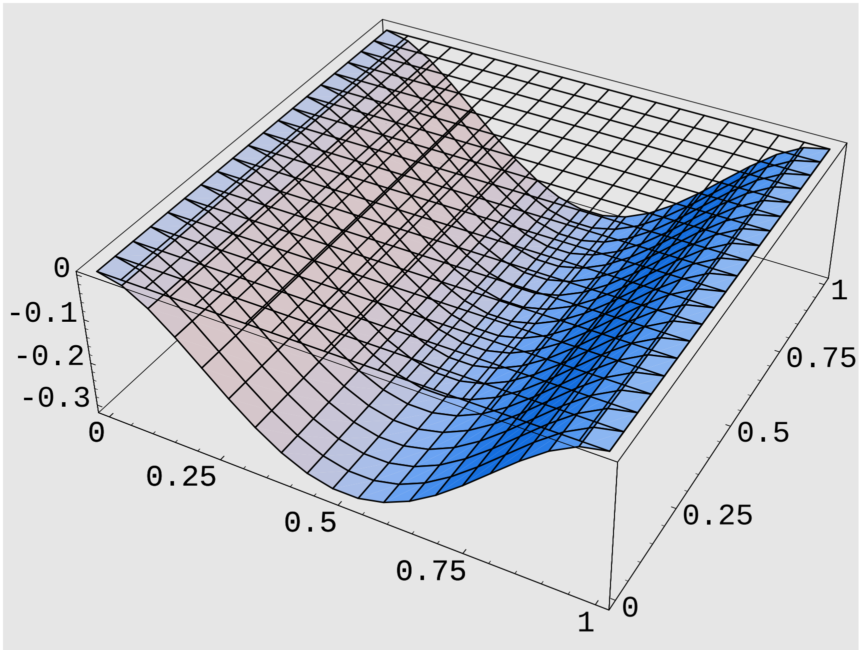

Figure 1: Large deformation of a plate.

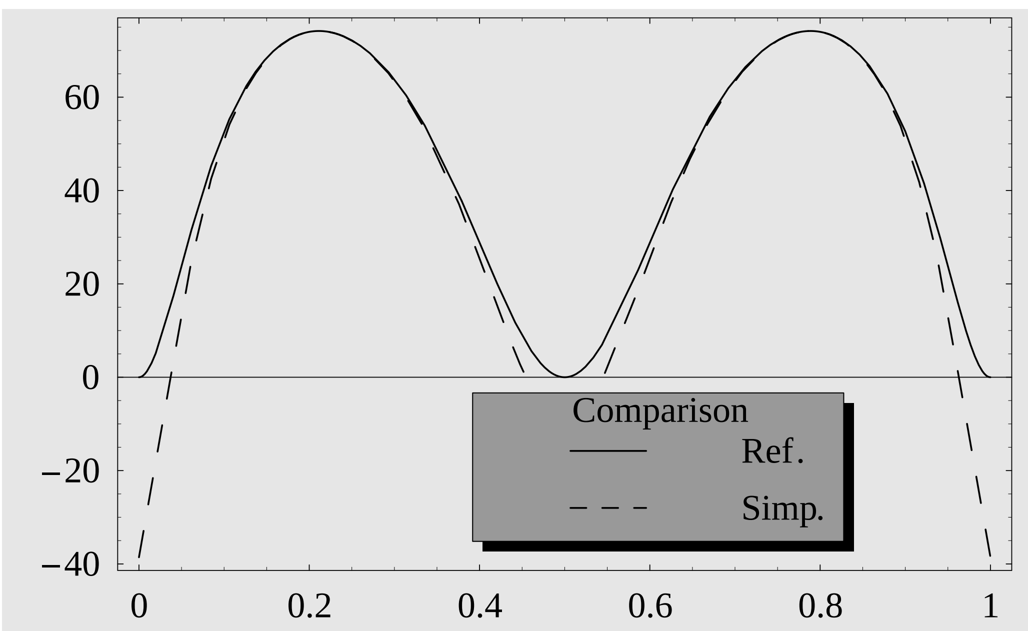

Figure 2: Physical component of axial force N_{11}^{-} for the refined and simplified approaches.

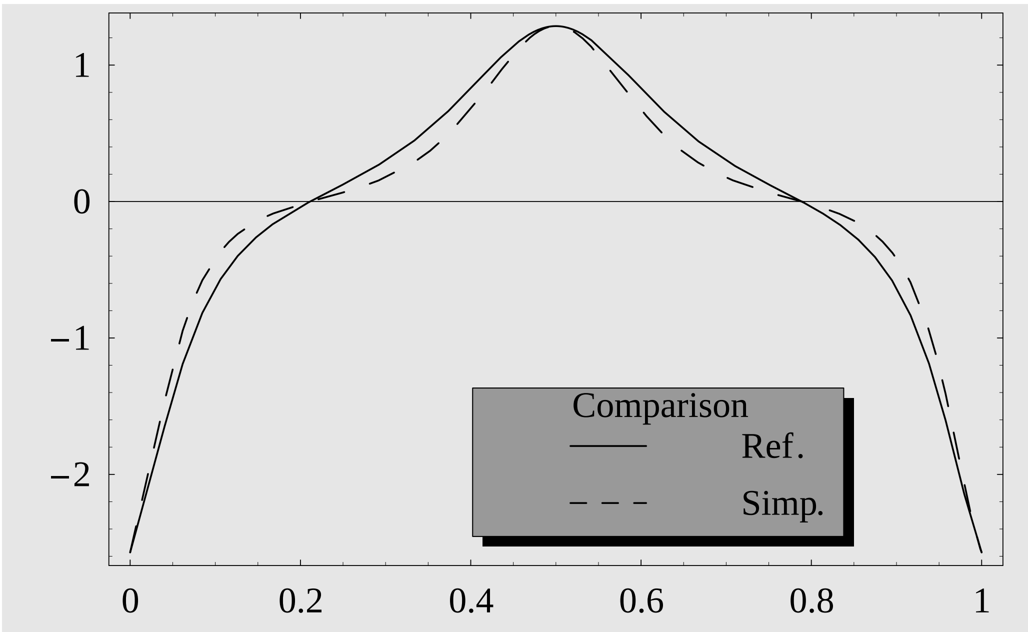

Figure 3: Physical component of bending moment M^{-}_{12} for the refined and simplified approaches.

6 Summary and final remarks

The presented topic is one of the range possibilities of the *MathTensor* applications for engineering purposes. This is one of the tasks which cannot be solved without computer algebra assistance.

Tensor analysis is a very elegant way of teaching problems of continuum mechanics but is regarded to be very difficult by most of students and engineers. Hand tensor calculations are tedious and prone to error [2]. They become more friendly with computer algebra system.

The presented topic is a part of wider studies developed. The more comprehensive material can be studied from the appropriate *Mathematica* notebooks. They can be obtained by sending email to the author.

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