SYMBOLIC COMPUTATIONS IN PROBLEMS OF MECHANICS

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In the report long-term experience of the authors on development of algorithms and automation of a research of complicated mechanical and controlled systems is considered. We have created specialized systems of computer algebra (CAS) and software packages "Dynamics", "Mechanic" possibility and the algorithms of which are described in [1-5 and other].

Currently we are developing software system [6-9] for solving mechanical problems based on CAS "Mathematica" [10]. The software allows to automatize, and consequently, essentially to speed up processes of modelling and dynamic analysis of complicated systems, to avoid errors at all stages of researches. The base of the algorithms, which are realized in these packages, was formed by classical methods of analytical mechanics and stability theory.

Our experience with symbolic computation packages allows us to conclude that CAS are perspective tool for researches in the field of theoretical mechanics.

1. Description of the models.

The software allows to automate generation of mathematical models (differential equations) of complex mechanical systems and electric circuits.

1.1. Model of the mechanical system. The mechanical system is a system of bodies S_j $(j = 1, \ldots, N)$, connected by one-two-three-degree joints, i.e. for every S_i there exists the point O_i ($O_i \in S_j$, $O_i \in S_i$, $i,\ j \in \{1, \ldots, N\}$, $i \neq j$) , or joints allowing translational displacements S_i relative to S_j (Fig. 1). The body S_1 is a carrier, S_i $(i = 2, \ldots, N)$ is carried. Let us associate a coordinate system Σ^1 to a rigid body S_1 . The starting point of that system will be in the point $O_1 \in S_1$. To a body S_i in a point $O_i \in S_i$ we will associate a coordinate system Σ^i . Angular position of Σ^i relative to Σ^j is described by a matrix $\alpha^{\,ji}$ with elements being the functions of generalized rotation coordinates.

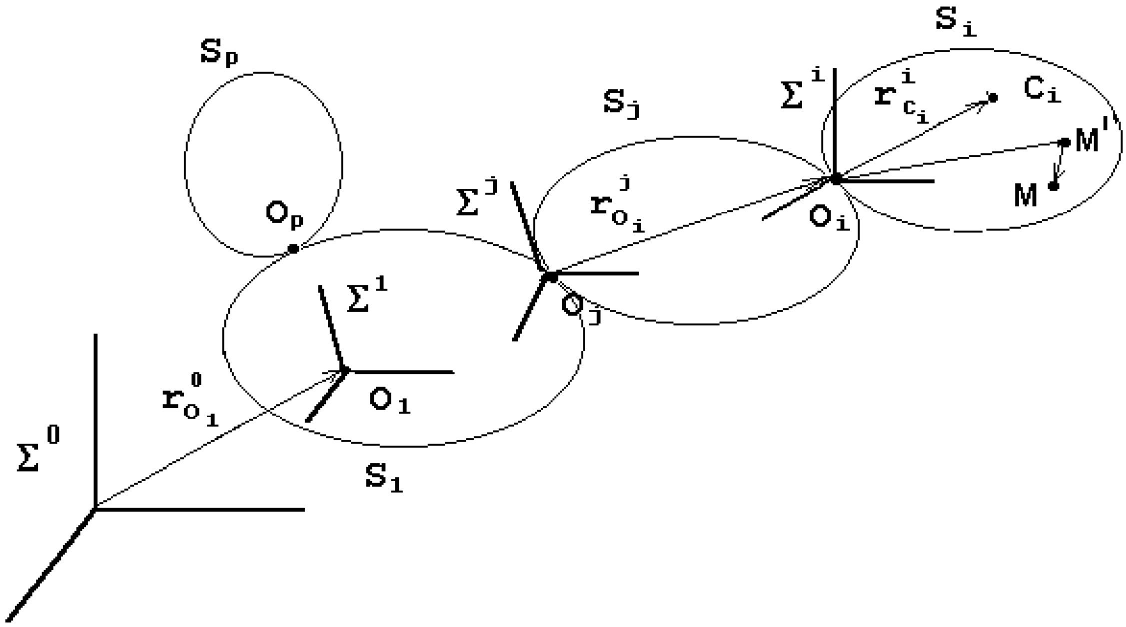

Figure 1.

The kinetic energy of a system is calculated as a sum of kinetic energies T_i for every body S_i , connected with S_j :

$$T_i = \frac{1}{2}\ M_i\ \mathbf{v}_{o_i}^2 + \frac{1}{2}\ \boldsymbol{\omega}_i \cdot \Theta^{o_i} \cdot \boldsymbol{\omega}_i + M_i\ [\ \mathbf{v}_{o_i} \times \boldsymbol{\omega}_i\ ] \cdot \mathbf{r}_{c_i}^i\ . \tag{1.1}$$

Here M_i is the mass of the body ;
$\mathbf{v}_{o_i}$ is absolute velocity of point O_i :

$$\mathbf{v}_{o_i} = \mathbf{v}_{o_j} + [\ \boldsymbol{\omega}_j \times \mathbf{r}_{o_i}^j\ ] + \mathbf{v}_{o_i}^j\ ; \tag{1.2}$$

$\boldsymbol{\omega}_i$ is absolute angular velocity of body :

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_j + \boldsymbol{\omega}_i^j \tag{1.3}$$

$\boldsymbol{\omega}_i^j$ is angular velocity of motion Σ^i relative to Σ^j ;
$\mathbf{v}_{o_i}^j$ is the relative velocity of point O_i , if S_i moves translationally (or freely) relative to S_j ;
$\mathbf{r}_{c_i}^i$ is the radius-vector of centre of mass of the body ;
Θ^{o_i} is the inertia tensor of the body relative to O_i .

Algorithm of calculation of kinetic energy and force function U (either in constant or in Newton's gravity field) for a system of interconnected rigid and deformable bodies is described in [9]. After calculation of these functions we can construct the lagrangian of the system :

$$L = T + U . \tag{1.4}$$

1.2. Description of an electric circuit. Electromechanical analogies [11] allow to use the same apparatus - methods of analytical mechanics - for describing and investigation of both mechanical systems and electric circuits.

Consider a linear electric circuit, in which resistors (R), inductors (L), capacitors (C), power sources of current (I) and voltage (E) are interconnected arbitrarily. For the purpose of describing the circuit let us choose a set of independent variables characterizing its state at any time moment. Such variables may be represented by either currents in the loops or voltages in the nodes or else currents and nodal voltages. Selection of a set of variables defines the structure of the equations.

One of the algorithms for constructing of the lagrangian for linear electric circuit is presented in [9].

Consider an algorithm of constructing the Lagrangian for the electric circuit. A set of currents will be used as the state variables. The computations are conducted in the two stages. On the 1st stage, the set of state variables of the electrical circuit is determined by the method of loop currents; on the 2d - the Lagrangian is computed.

The general idea of the method of loop currents consists in decomposing the electrical circuit into independent geometric loops and assigning the current in each of the loops. This problem may be solved via finding a set of fundamental cycles in a graph.

Let us represent the graph as the list of vertex pairs $\{n_k,\ n_m\}$ that describe corresponding edges.

When denoting by $\dot{q}_i$ the current in the $i-$th loop and considering it positive in the edge $\{n_k,\ n_m\}$, when $n_k < n_m$ and otherwise negative, we shall find the magnitude of the current in each branch as the sum or the difference of the corresponding loop currents.

According to [11] the kinetic energy T, the force function U, the Rayleigh function $\tilde{R}$ for the linear electrical circuit has the form :

$$T = \frac{1}{2} L_{ij}\dot{q}_i\dot{q}_j \ , \quad U = -\frac{1}{2}\frac{q_i q_j}{C_{ij}} \ , \quad \tilde{R} = \frac{1}{2} R_{ij}\dot{q}_i\dot{q}_j \ , \ \ (\ i,j = 1,\ldots,k \ ),$$

where $L_{ij}-$ inductance, $C_{ij}-$ capacitance, $R_{ij}-$ resistance, q_i- charge.

From now on summation is executed with respect to the index which occurs twice.

In electric circuits it is often the case that

$$\det \| \frac{\partial^2 L(q,\dot{q})}{\partial \dot{q}_i \partial \dot{q}_j} \| = 0 \ , \qquad (\ i,j = 1,\ldots,k \ ) \ .$$

Therefore it is not always possible to execute standard transformation of generalized velocities to Hamilton variable. Partial transformation is in this case carried out. The singularities of a conclusion of the Hamilton's equations for such situations are considered in [12].

1.3. Basic problems. Having calculated a lagrangian (1.4) of the system after geometric description of the body system or electric circuit, with the help of software we can obtain the following results :
1. differential equations of motion in second Lagrange form in generalized coordinates ;
2. differential equations of motion in Euler-Lagrange form in quasicoordinates ;
3. differential equations of motion in Hamilton form in generalized coordinates and impulses ;
4. differential equations of motion in Routh form in generalized coordinates, velocities and impulses ;
5. first integrals of specific form (linear with respect to velocities, cyclicin particular, and quadratic with respect to velocities) for nonlinear motion equations or existence conditions for these integrals ;
6. equations of steady motions and invariant manifolds of steady motions and solutions of these equations;
7. sufficient conditions of stability of steady motions using algorithm of Routh-Lyapunov theorem ;
8. first approach equations in the neighbourhood of a given (or found) motion ;
9. characteristical equation for linear systems ;
10. conditions of stability of a given motion by first approach equations (Routh-Hurwitz criterion, Lineard-Chipart criterion, criterion of existence of real and negative roots of the polynomial, calculation of first Lyapunov value, criterion of the safety and danger of stability border, etc.) ;
11. Lyapunov functions for investigation of stability depending on forces structure in equations with separated linear part ;
12. Solving parametrical systems of inequalities.

All calculations are done in symbolic form. If necessary it's possible to substitute numerical values after constructing of motion equations and obtaining necessary conditions of stability. There are no algorithmic limitations on the number of degrees of freedom k .

As follows from the problems listed above, our packages are different from other software projects because they contain big algorithmic modules allowing to do a qualitative investigation of generated differential equations.

Description of some algorithms for solving abovementioned problems is presented in [9]. Here we will present a short description of algorithms of qualitative investigation using Lyapunov function method.

2. Stability of the steady motions.

A question of separating stable steady motions of the system always was actual. Algorithms based on Routh-Lyapunov theorem [13] are especially suitable for implementation in CAS. In accordance with the latter theorem we come to the problem of conditional extremum.

Let for the system of ordinary differential equations

$$\dot{x} = X(x) \ , \qquad x \in R^k$$

some number of first integrals

$$V_0(x) = c_0 \ , \quad V_1(x) = c_1 \ , \ \dots \ , \quad V_m(x) = c_m \ .$$

be known. Having chosen these integrals as the basis, we define the linear space of system's integrals over R :

$$K_J = \sum_{j=1}^{n} \lambda_{i_j} V_{i_j}(x) \ , \quad J = \{i_0, \dots, i_n\} \in (0, 1, \dots, m) \ . \tag{2.1}$$

Let us find steady motions (invariant manifolds of steady motions - IMSM) as solutions of the system of equations

$$\frac{\partial K}{\partial x_i} = f_i(x_1, \dots x_k, \lambda_{i_1}, ..., \lambda_{i_n}) = 0 \ , \qquad (i = 1, \dots \ , \ k)$$

$$V_{i_1}(x) = c_{i_1} \ , \ \dots \ , \ V_{i_n}(x) = c_{i_n} \ .$$

2.1. Algorithm of Routh-Lyapunov theorem in generalized coordinates. Let us assume that lagrangian (1.4) in a conservative system has a form :

$$L = \frac{1}{2} a_{ij}(q)\dot{q}_i\dot{q}_j \ + \ a_i(q)\dot{q}_i + a_0(q) \ , \quad (\ i, j = 1, \dots, k \ ) \tag{2.2}$$

where k is a number of degrees of the freedom of the system ; q, $\dot{q}$ are generalized coordinates and velocities; $a_0(q) = T_0(q) + U(q)$, $T_0(q)$ fraction of kinetic energy that does not depend from velocities ; $a_{ij} = a_{ji}$. Differential equations of the system motion allow Jacoby integral $H \ = 1/2a_{ij}(q)\dot{q}_i\dot{q}_j - a_0(q) \ = h$, $(\ i, j = 1, \dots, k \ )$. Let us suppose that there exists an integral W_1 quadratic with respect to velocities, noncyclic linear integral W_2 with respect to velocities, and cyclic integrals $V_\alpha = \partial L / \partial \dot{q}_\alpha = c_\alpha$, $(\alpha = 1, \dots, m)$. Maximal linear bundle of integrals (2.1):

$$K = H + \mu_1 W_1 + \mu_2 W_2 + \lambda_\alpha V_\alpha \ , \qquad (\alpha = 1, \dots, \ m) \ . \tag{2.3}$$

Necessary extremal conditions of K have the form:

$$\frac{\partial K}{\partial \dot{q}_i} = 0 \ , \quad (\ i = 1, \ldots, k \ ) \ ; \qquad \frac{\partial K}{\partial q_j} = 0 \ , \quad (\ j = m+1, \ldots, k \ ) \ .$$

First group of equations is linear with respect to $\dot{q}_i$ and depending on the determinant of that system can have either singular or nonsingular solution for $\dot{q}_i$. In latter case we will have degenerated solutions. After substituting a solution for $\dot{q}_i$ of second group of equations we will get an equation system, generally nonlinear, for coordinates q_j $(j = m+1, \ldots, k)$ and λ_α , μ_1 , μ_2 . If the solution is steady for different first integrals, we will call it special. Investigating stability of degenerated and special steady motions is discussed in [14].

Let in a bundle (2.3) $\mu_1 = \mu_2 = 0$.

$$V_\alpha = a_{\alpha i}\dot{q}_i + a_\alpha = const \ , \quad (\ \alpha = 1, \ldots, m; \ \ i = 1, \ldots, k \ ) \ .$$

Coefficient matrix a_{ij} is not degenerated. Thus equations $\partial K / \partial \dot{q}_j = a_{ij}\dot{q}_i + \lambda_\alpha a_{\alpha j} = 0$ $(j = 1, \ldots, k)$ have singular family of solutions (λ_α is a family parameter):

$$\dot{q}_i = -\delta_{i\alpha}\lambda_\alpha \ , \quad (\ i = 1, \ldots, k \ ; \ \alpha = 1, \ldots, m) \ , \tag{2.4}$$

where $\delta_{i\alpha}$ is a Croneker symbol.

After substituting a solution (2.4) equations $\partial K / \partial q_j = 0$ are transformed to the form:

$$-\frac{1}{2}\lambda_\alpha\lambda_\nu\frac{\partial a_{\alpha\nu}}{\partial q_j} + \lambda_\alpha\frac{\partial a_\alpha}{\partial q_j} - \frac{\partial a_0}{\partial q_j} = 0 \ , \quad (\ j = m+1, \ldots, k \ ; \ \nu, \ \alpha = 1, \ldots, \ m \ ). \tag{2.5}$$

These relations define constant values of positional coordinates in steady motion with any fixed set of λ_α .

Sufficient stability conditions of selected motions we cat obtain as conditions of sign-definitiveness of the second variation of $\delta^2 K$ of function K in the neighborhood of the motion in question with constant values of cyclic integrals:

$$(\delta^2 K)_{\delta V_\alpha = 0} >> 0 \qquad (\ \alpha = 1, \ldots, \ m \ ) \ ;$$

$$\delta^2 K = \frac{1}{2}(a_{ij})_0 \delta\dot{q}_i \delta\dot{q}_j + \frac{1}{2}c_{lp}\delta q_l \delta q_p \ ; \qquad \delta V_\alpha = (a_{\alpha i})_0 \delta\dot{q}_i + b_{\alpha l}\delta q_l = 0 \ .$$

Here

$$c_{lp} = \left(-\frac{\partial^2 a_0}{\partial q_l \partial q_p} + \lambda_\alpha \frac{\partial^2 a_\alpha}{\partial q_l \partial q_p} + \frac{1}{2}\lambda_\nu\lambda_\alpha\frac{\partial^2 a_{\alpha\nu}}{\partial q_l \partial q_p} \right)_0 \ ; \quad b_{\alpha l} = \left(-\lambda_\nu \frac{\partial a_{\nu\alpha}}{\partial q_l} + \frac{\partial a_\alpha}{\partial q_l} \right)_0 \ ;$$

$$(\ i, \ j = 1, \ldots, k \ ; \ l, \ p = m+1, \ldots, k \ ; \ \nu, \ \alpha = 1, \ldots, m \ ) \ ,$$

$(\ldots)_0$ are coefficients of linear and quadratic forms calculated for steady motions.

Determinant construct for the solution of the problem $\delta^2 K >> 0$ with $\delta V_\alpha = 0$ $(\alpha = 1, \ldots, m)$ with simple transformations can be changed to form :

$$\Delta^* = det\|(a_{ij})_0\|\Delta = det\|(a_{ij})_0\|(-1)^m \begin{vmatrix} c_{lp} & | & b_{\nu p} \\ ------- & | & ------ \\ b_{\alpha p} & | & -(a_{\nu\alpha})_0 \end{vmatrix}$$

$$(\ i, \ j = 1, \ldots, k \ ; \ l, \ p = m+1, \ldots, k \ ; \ \nu, \ \alpha = 1, \ldots, m \ ) \ . \qquad (2.6)$$

If $\Delta > 0$ and all determinants, obtained from Δ by removing $(k-m)-$th, $(k-m-1)-$th, $\ldots$, rows and columns are positive, then $\delta^2 K >> 0$ when $\delta V_\alpha = 0$. These inequalities give us sufficient conditions of steady motions stability with respect to all generalized velocities and positional coordinates. These sufficient conditions are close to necessary ones. Indeed, characteristic equation of the first approach differential equations system of perturbed motion in the neighborhood of our steady motion have the form:

$$n^{2m} \begin{vmatrix} (a_{\alpha\nu})_0 & | & (a_{\alpha l})_0 n + b_{\alpha l} \\ --------- & | & ---------- \\ (a_{p\alpha})_0 n - b_{\nu p} & | & (a_{pl})_0 n^2 + \gamma_{pl} n + c_{pl} \end{vmatrix} = 0 \ , \qquad (2.7)$$

$$\text{where} \qquad \gamma_{il} = -\gamma_{li} = -\lambda_\alpha \left(\frac{\partial a_{i\alpha}}{\partial q_l} - \frac{\partial a_{l\alpha}}{\partial q_i} \right)_0 \ ,$$

$$(\ i = 1, \ldots, k \ ; \ l, \ p = m+1, \ldots, k \ ; \ \nu, \ \alpha = 1, \ldots, m \ ) \ .$$

Free with respect to n member of the determinant (2.7) is equal to determinant (2.6) . If $\Delta < 0$, then steady motion in question is unstable. Thus one of sufficient conditions coincide (without border) with necessary.

If a system is under influence of the forces with generalized potential $\Pi(q, \dot{q}, t) = \Pi_0(q, t) + b_i(q, t)\dot{q}_i$, then generalized forces

$$Q_i = \frac{d}{dt}\frac{\partial \Pi}{\partial \dot{q}_i} - \frac{\partial \Pi}{\partial q_i} = -\frac{\partial \Pi_0}{\partial q_i} + \frac{\partial b_i}{\partial t} + \dot{q}_j \left(\frac{\partial b_i}{\partial q_j} - \frac{\partial b_j}{\partial q_i} \right) \ , \quad (\ i, \ j = 1, \ldots, \ k \ )$$

can be divided to potential, positional and gyroscopic.

If the system allows cyclic integrals and energy integral, then all the formulae above are applicable if we substitute a_i to $d_i = (a_i - b_i)$, a_0 to $(T_0 - \Pi_0)$,

$$\gamma_{il} = -\gamma_{li} = -\lambda_\alpha \left(\frac{\partial a_{i\alpha}}{\partial q_l} - \frac{\partial a_{l\alpha}}{\partial q_i} \right)_0 + \left(\frac{\partial d_l}{\partial q_i} - \frac{\partial d_i}{\partial q_l} \right)_0 \ ,$$

$$(\ i=1,\ldots,\ k\ ; \quad l=m+1,\ldots,k\ ; \quad \alpha=1,\ldots,m\ )$$

2.2. Algorithm of Routh-Lyapunov theorem in quasicoordinates. Let conservative mechanical system with k degrees of freedom has the lagrangian (1.4) :

$$L=\frac{1}{2}A_{ij}(q)\Omega_i\Omega_j+U(q)\ ,$$

expressed via generalized coordinates q_i and quasivelocities

$$\Omega_i=a^*_{ij}(q)\dot{q}_j\ , \qquad (\ i,\ j=1,\ldots,k\ )\ . \tag{2.8}$$

Let us assume that for coordinates $q_1,\ldots,q_m$ existence conditions of cyclic integrals

$$V_p=\frac{\partial T}{\partial\Omega_i}a^*_{ip}(q)=const\ , \qquad (\ i=1,\ldots,k\ ,\ p=1,\ldots,m\ ) \tag{2.9}$$

are met.

Let us construct a bundle of first integrals $K=H+\lambda_pV_p$, $(p=1,\ldots,m\ )$, where $H=T-U$ is integral of energy, λ_p are Lagrange constant multiplicators in the problem of conditional extremum.
Stationarity equations K with respect to quasivelocities give singular relation between Ω_i and λ_p:

$$\Omega_i=-\lambda_pa^*_{ip}\ , \qquad (\ i=1,\ldots,k\ ,p=1,\ldots,m\ )\ . \tag{2.10}$$

Stationarity equations K with respect to positional coordinates after substituting relations (2.10) are being written out in the form:

$$\frac{\partial}{\partial q_\tau}(U+\frac{1}{2}A_{ji}\lambda_h\lambda_pa^*_{jh}a^*_{ip})=0\ , \qquad (\ h,p=1,\ldots,m\ )\ . \tag{2.11}$$

From equations system (2.11) for each set of parameters λ_α $(\alpha=1,\ldots,m)$ we can define a set of positional coordinate values for elements of steady motion family. Sufficient conditions of steady motions stability with respect to Ω_i $(i=1,\ldots,k)$ and positional coordinates $(k-m)$ can be obtained as conditions of sign-definitiveness of second variation of function K in the neighborhood of selected steady motion

$$\delta^2K=\frac{1}{2}(A_{ij})_0\delta\Omega_i\delta\Omega_j+(b_{i\tau})_0\delta\Omega_i\delta q_\tau+\frac{1}{2}(c_{\tau\eta})_0\delta q_\tau\delta q_\eta$$

on linear manifold defined by variations of cyclic integrals

$$\delta V_h=(d_{hi})_0\delta\Omega_i+(e_{h\tau})_0\delta q_\tau=0\ ,$$

where

$$U^* = U + \frac{1}{2} A_{ij} \lambda_h \lambda_p a^*_{ih} a^*_{jp} \ ; \qquad c_{\tau\eta} = A_{ij} \lambda_h \lambda_p \frac{\partial a^*_{jp}}{\partial q_\tau} \frac{\partial a^*_{ih}}{\partial q_\eta} - \frac{\partial^2 U^*}{\partial q_\tau \partial q_\eta} \ ;$$

$$b_{i\tau} = A_{ij} \lambda_p \frac{\partial a^*_{ip}}{\partial q_\tau} \ ; \qquad d_{hi} = A_{ij} a^*_{jh} \ ; \qquad e_{h\tau} = -\lambda_p a^*_{jp} \left(\frac{\partial A_{ij}}{\partial q_\tau} a^*_{ih} + A_{ij} \frac{\partial a^*_{ih}}{\partial q_\tau} \right),$$

$$(\ i, \ j = 1, \ldots, k \ ; \ \ h, \ p = 1, \ldots, m \ ; \ \ \tau, \ \eta = m+1, \ldots, k \ ) \ .$$

Maximal determinant, constructed for solving a problem $(\delta^2 K)_{\delta V_h = 0} \gg 0$ by big number of elementary transformations can be changed to a form $\Delta_1^* = det \parallel A \parallel * \Delta_1$. Here for matrix of coefficients A_{ij} we have $det \parallel A \parallel > 0$ because of positive definitiveness of quadratic form of kinetic energy,

$$\Delta_1 = det \begin{pmatrix} (W_{11})_0 & \ldots & (W_{1,m})_0 & (G_{1,m+1})_0 & \ldots & (G_{1,k})_0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (W_{m,1})_0 & \ldots & (W_{m,m})_0 & (G_{m,m+1})_0 & \ldots & (G_{m,k})_0 \\ & & & & & \\ (-G_{m+1,1})_0 & \ldots & (-G_{m+1,m})_0 & (F_{m+1,m+1})_0 & \ldots & (F_{m+1,k})_0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (-G_{k,1})_0 & \ldots & (-G_{k,m})_0 & (F_{k,m+1})_0 & \ldots & (F_{k,k})_0 \end{pmatrix} \tag{2.12}$$

Sufficient conditions of stability will be requirements of positiveness of main diagonal minors of matrix (2.12), starting from $(m+1)-$th and ending with $k-$th order. Here elements of the determinant are expressed as

$$G_{\tau h} = -\frac{\partial}{\partial q_\tau} (A_{ji} \lambda_p a^*_{jp} a^*_{ih}) \ ; \quad F_{\tau\eta} = -\frac{\partial^2}{\partial q_\tau \partial q_\eta} (U + \frac{1}{2} A_{ji} \lambda_h \lambda_p a^*_{jh} a^*_{ip}) \ ; \quad W_{ph} = A_{ij} a^*_{jh} a^*_{ip}$$

$$(\ i, \ j = 1, \ldots, k \ ; \ \ h, \ p = 1, \ldots, m \ ; \ \ \tau, \ \eta = m+1, \ldots, k \ ) \ .$$

As in item 2.1, stability conditions obtained with this method are close to necessary, because one of sufficient condition does match to necessary (without a boundary).

Systems with first integrals are critical in Lyapunov sense. Near the critical case numerical methods are not reliable, thus it's better to use methods of computer algebra. One of it's specific abilities is reaching a precise analytical zero. Verification of these conditions and explicit form of integrals is possible only with the help of CAS. The CAS should be able to do complex trigonometrical transformations.

For obtaining first integrals we use differential consequences and invariant correlations. For equations of Lagrange's and Euler-Lagrange's form algorithms of constructing differential consequences and invariant relations are described in [6,8]. For linear systems the problem of constructing first integrals of linear and quadratic forms is reduced to solution of linear algebraic equations.

3. Examples.

Let us discuss several examples of using described algorithms.

3.1. Dynamic analysis of gyroscopic frame. Let us consider gyroscopic frame, placed in Newtonean central field of forces. The mechanical system consists of a carrying frame (a body with a fixed point) and two identical connected two-degree gyroscopes, symmetrically installed in a frame (Fig. 2). Masses and the moments of inertia of housing of gyroscopes are neglected.

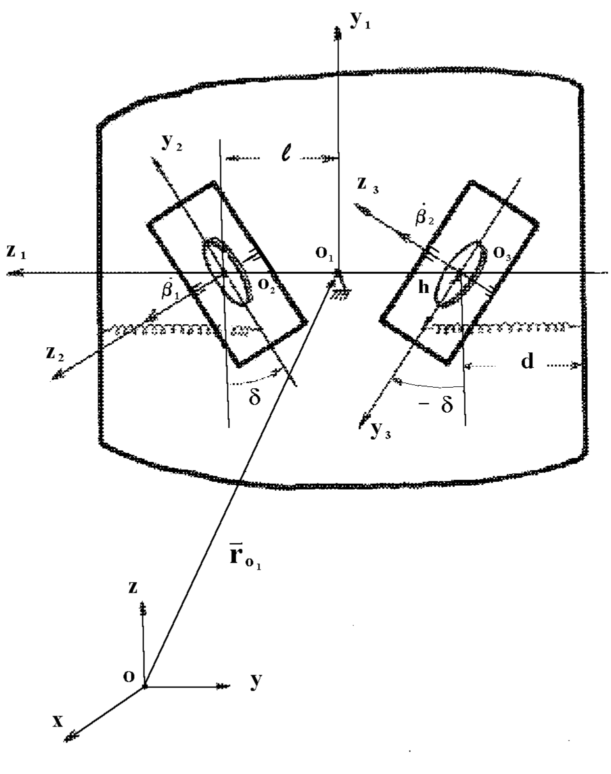

Figure 2.

Before exposition of input information about the system we note that the matrix of rotation α^{ij} and relative angular velocities of bodies $\boldsymbol{\omega}_i^j$ are not required to be introduce. They are calculated automatically on specific "sequence of rotations". Hereby we specify :

a) Number of axis of rotation (one of numbers 0, 1, 2, 3), i.e. 1 - the rotation is carried out around the axis Ox_i ; 2 - around the axis Oy_i ; 3 - around the axis Oz_i ; 0 -

there is no rotation;
b) angle of rotation.

Input data :

Number of bodies in the system : $N = 5$.
Body 1 is a frame. M_1 is mass ;

$$\mathbf{r^0_{o_1}} = \begin{pmatrix} 0 \\ 0 \\ R \end{pmatrix} ; \quad \mathbf{r^1_{c_1}} = \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} ; \quad \mathbf{v^0_{o_1}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ; \quad \boldsymbol{\omega}_1 = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} ;$$

$$\text{the inertia tensor :} \qquad \Theta^{o_1}_* = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & C_1 \end{pmatrix}$$

"sequence of rotations" : $3 \ , \ \psi \ ; \ 1 \ , \ \theta \ ; \ 3 \ , \ \varphi$.
Body 2 is the housing of the first gyroscope, it is connected to the frame. $M_2 = 0$;

$$\mathbf{r^1_{o_2}} = \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix} ; \quad \mathbf{r^2_{c_2}} = \mathbf{v^1_{o_2}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ; \quad \Theta^{o_2}_* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

"sequence of rotations" : $1 \ , \ \delta$.
Body 3 is the rotor of the first gyroscope, it is connected to the second body. m is mass of a rotor;

$$\mathbf{r^2_{o_3}} = \mathbf{r^3_{c_3}} = \mathbf{v^2_{o_3}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ; \qquad \Theta^{o_3}_* = \begin{pmatrix} A_3 & 0 & 0 \\ 0 & A_3 & 0 \\ 0 & 0 & C_3 \end{pmatrix}$$

"sequence of rotations" : $3 \ , \ \beta_1$.
Body 4 is the housing of the second gyroscope, it is connected to a frame. $M_4 = 0$;

$$\mathbf{r^1_{o_4}} = \begin{pmatrix} 0 \\ 0 \\ -l \end{pmatrix} ; \quad \mathbf{r^4_{c_4}} = \mathbf{v^1_{o_4}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ; \qquad \Theta^{o_4}_* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

"sequence of rotations" : $1 \ , \ -\delta$.
Body 5 is the rotor of the second gyroscope, it is connected to the fourth body. m is mass of a rotor;

$$\mathbf{r^4_{o_5}} = \mathbf{r^5_{c_5}} = \mathbf{v^4_{o_5}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ; \qquad \Theta^{o_5}_* = \begin{pmatrix} A_3 & 0 & 0 \\ 0 & A_3 & 0 \\ 0 & 0 & C_3 \end{pmatrix}$$

"sequence of rotations" :   3 , β_2 .

<u>Program output :</u>

Matrices of rotation :

$$\alpha^{01} = \begin{pmatrix} (\cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\theta) & (\cos\varphi\sin\psi + \sin\varphi\cos\psi\cos\theta) & \sin\theta\sin\varphi \\ (-\sin\varphi\cos\psi - \cos\varphi\cos\theta\sin\psi) & (\cos\varphi\cos\theta\cos\psi - \sin\varphi\sin\psi) & sin\theta\cos\varphi \\ \sin\theta\sin\psi & -\cos\psi\sin\theta & \cos\theta \end{pmatrix};$$

$$\alpha^{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & \sin\delta \\ 0 & -\sin\delta & \cos\delta \end{pmatrix}; \quad \alpha^{23} = \begin{pmatrix} \cos\beta_1 & \sin\beta_1 & 0 \\ -\sin\beta_1 & \cos\beta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$\alpha^{14} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & -\sin\delta \\ 0 & \sin\delta & \cos\delta \end{pmatrix}; \quad \alpha^{45} = \begin{pmatrix} \cos\beta_2 & \sin\beta_2 & 0 \\ -\sin\beta_2 & \cos\beta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Relative angular velocities :

$$\boldsymbol{\omega}_2^1 = \begin{pmatrix} \dot\delta \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{\omega}_3^2 = \begin{pmatrix} 0 \\ 0 \\ \dot\beta_1 \end{pmatrix}; \quad \boldsymbol{\omega}_4^1 = \begin{pmatrix} -\dot\delta \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{\omega}_5^4 = \begin{pmatrix} 0 \\ 0 \\ \dot\beta_2 \end{pmatrix}.$$

The kinetic energy (1.1)–(1.3) of the system :

$$T = \frac{1}{2}[A\Omega_1^2 + B\Omega_2^2 + C\Omega_3^2 + 2A_3\dot\delta^2 + C_3(\dot\beta_1^2 + \dot\beta_2^2)] + C_3[\Omega_2\sin\delta(\dot\beta_2 - \dot\beta_1) + \Omega_3\cos\delta(\dot\beta_1 + \dot\beta_2)]$$

where $A = A_1 + 2ml^2 + 2A_3$; $C = C_1 + 2A_3\sin^2\delta + 2C_3\cos^2\delta$;
$B = B_1 + 2ml^2 + 2A_3\cos^2\delta + 2C_3\sin^2\delta$.

$\Omega_1, \ \Omega_2, \ \Omega_3$ are the projections of absolute angular velocity of the frame on the axes $x_1, \ y_1, \ z_1$. They are selected as quasivelocities :

$$\Omega_1 = \dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi; \quad \Omega_2 = \dot\psi\sin\theta\cos\varphi - \dot\theta\sin\varphi; \quad \Omega_3 = \dot\psi\cos\theta + \dot\varphi .$$

The remaining quasivelocities coincide with generalized velocities $\dot\delta$, $\dot\beta_1$, $\dot\beta_2$.
Approximate force function U in Newtonean field of gravitation to the fixed center O

$$U = \mu \, [\, \frac{1}{6}(A_1 + B_1 + C_1) - \frac{1}{2}(A_1\gamma_1^2 + B_1\gamma_2^2 + C_1\gamma_3^2) + \frac{1}{3}(2A_3 + C_3 - ml^2) -$$
$$-A_3(\gamma_1^2 + \gamma_2^2\cos^2\delta + \gamma_3^2\sin^2\delta) - C_3(\gamma_2^2\sin^2\delta + \gamma_3^2\cos^2\delta) + ml^2\gamma_3^2 \,] -$$
$$-M_1 \mathrm{g}(x_c\gamma_1 + y_c\gamma_2 + z_c\gamma_3) + (M_1 + 2m)\frac{\nu}{R} + const .$$

Here $\gamma_1 = \sin\varphi \sin\theta$, $\gamma_2 = \sin\theta \cos\varphi$, $\gamma_3 = \cos\theta$ are direction cosines of axis Oz with axes $x_1,\ y_1,\ z_1$; ν is the constant of gravitation ;

$$\mathrm{g} = \frac{\nu}{R^2} \text{ is acceleration of constant forse of gravity in point } O_1 \ ; \quad \mu = \frac{3\nu}{R^3} \ .$$

The force function of elastic forces of the spring device :

$$U_*(\delta) \ = \ -2\,\chi\ \sin^2\frac{\delta}{2} \ , \qquad \text{where} \qquad \chi \ = \ const \ .$$

The software program found four first integrals of motions equations :
integral of an energy H and three cyclic integrals (2.9) (with respect to ignorable coordinates $\psi,\ \beta_1,\ \beta_2$)

$$H \ = \ T \ - \ U \ - \ U_* = c_0 \ ,$$

$$V_1 = \frac{\partial T}{\partial \Omega_1}\gamma_1 + \frac{\partial T}{\partial \omega_2}\gamma_2 + \frac{\partial T}{\partial \omega_3}\gamma_3 = c_1 \ , \qquad V_2 = \frac{\partial T}{\partial \dot\beta_1} = c_2 \ , \qquad V_3 = \frac{\partial T}{\partial \dot\beta_2} = c_3 \ , \qquad (3.1)$$

where $c_0,\ c_1,\ c_2,\ c_3$ are integral constants.

According to number of ignorable coordinates the program renumerates generalized coordinates that were input before : $q_1 \ = \ \psi$, $q_2 \ = \beta_1$, $q_3 \ = \ \beta_2$, $q_4 \ = \ \delta$, $q_5 \ = \ \theta$, $q_6 \ = \ \varphi$.

Using the algorithm (item 2.2) the software program automatically constructs a bundle of integrals :

$$K = H + \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3 \ .$$

Value (2.10) of quasivelocities in steady motion:

$$\Omega_1 = -\lambda_1 \sin\theta_0 \sin\varphi_0 \ , \quad \Omega_2 = -\lambda_1 \sin\theta_0 \cos\varphi_0 \ , \quad \Omega_3 = -\lambda_1 \cos\theta_0 \ ,$$

$$\dot\delta = 0 \ , \quad \dot\beta_1 = -\lambda_2 \ , \quad \dot\beta_2 = -\lambda_3 \ . \qquad (3.2)$$

Constant values of positional coordinates $\theta_0,\ \varphi_0,\ \delta_0$ in steady motion fulfill the equations (2.11) :

$$\frac{\partial K}{\partial \delta} = \chi \sin\delta_0 + \lambda_1 C_3 \ [\ (\lambda_2 - \lambda_3)\cos\varphi_0 \cos\delta_0 \sin\theta_0 + (\lambda_2 + \lambda_3)\sin\delta_0 \cos\theta_0 \ ] +$$

$$+ \ \Lambda(A_3 - C_3)\sin 2\delta_0(\sin^2\theta_0 \cos^2\varphi_0 - \cos^2\theta_0) = 0$$

$$\frac{\partial K}{\partial \theta} = -\frac{\Lambda}{2}\sin 2\theta_0 \ [\ \sin^2\varphi_0(A - 2ml^2) + \cos^2\varphi_0(B - 2ml^2) - (C - 2ml^2)\ ] +$$

$$+ \ \lambda_1 C_3[(\lambda_2 - \lambda_3)\cos\varphi_0 \cos\theta_0 \sin\delta_0 + (\lambda_2 + \lambda_3)\sin\theta_0 \cos\delta_0] +$$

$$+ \ M_1 \mathrm{g}[\cos\theta_0(x_c \sin\varphi_0 + y_c \cos\varphi_0) - z_c \sin\theta_0] = 0$$

$$\frac{\partial K}{\partial \varphi} = \sin\theta_0 \ [\ \lambda_1 C_3(\lambda_3 - \lambda_2) \sin\varphi_0 \sin\delta_0 + \frac{\Lambda}{2}(B - A) \sin 2\varphi_0 \sin\theta_0 +$$

$$+ \ M_1 \mathrm{g}(x_c \cos\varphi_0 - y_c \sin\varphi_0) \ ] = 0 \tag{3.3}$$

Here λ_1 is equal to angular velocity of precession of frame ; λ_2 , λ_3 are equal to angular velocities of rotors; constant $\Lambda = \lambda_1^2 - \mu$.

From all the family of steady motions, determined by equations system (3.1)–(3.3), let us investigate stability of the following solution :

$$\theta_0 = \frac{\pi}{2} \ , \quad \varphi_0 = 0 \ , \quad \delta_0 = 0 \ , \quad \lambda_3 = \lambda_2 \ . \tag{3.4}$$

Existence conditions for a motion (3.4) are :

$$x_c = 0 \ , \qquad 2C_3\lambda_1\lambda_2 - M_1 \mathrm{g} z_c = 0 \ . \tag{3.5}$$

Matrix (2.12) on the solution (3.4)–(3.5) has form :

$$\begin{pmatrix} (w_{11})_0 & 0 & 0 & 0 & 2C_3\lambda_2 & 0 \\ 0 & C_3 & 0 & \lambda_1 C_3 & \lambda_1 C_3 & 0 \\ 0 & 0 & C_3 & -\lambda_1 C_3 & \lambda_1 C_3 & 0 \\ 0 & -\lambda_1 C_3 & \lambda_1 C_3 & (f_{44})_0 & 0 & 0 \\ -2C_3\lambda_2 & -\lambda_1 C_3 & -\lambda_1 C_3 & 0 & (f_{55})_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (f_{66})_0 \end{pmatrix}$$

where

$$(w_{11})_0 = B_1 + 2ml^2 + 2A_3 \ ; \qquad (f_{44})_0 = \chi + 2\Lambda(A_3 - C_3)$$

$$(f_{55})_0 = \Lambda[B_1 - C_1 + 2(ml^2 + A_3 - C_3)] - M_1 \mathrm{g} y_c \ ; \qquad (f_{66})_0 = \Lambda(B_1 - A_1) - M_1 \mathrm{g} y_c \ .$$

Positiveness of main diagonal minors of 4th, 5th and 6th orders of that matrix will define sufficient conditions of stability of the solution (3.4)–(3.5) :

$$(f_{44})_0 + 2\lambda_1^2 C_3 \ > 0 \ ; \qquad (w_{11})_0[\ 2\lambda_1^2 C_3 + (f_{55})_0 \ ] + 4C_3^2\lambda_2^2 \ > 0 \ ; \qquad (f_{66})_0 \ > 0 \ .$$

Later conditions are met, for example, when

$$A_3 > C_3 \ , \qquad \lambda_1^2 > \mu \ , \qquad y_c < 0 \ , \qquad B_1 > C_1 \ , \qquad B_1 > A_1 \ .$$

All the expressions obtained by computer have the symbolic form.

3.2. The modelling of the circuit. Let us consider the circuit in Fig. 3.

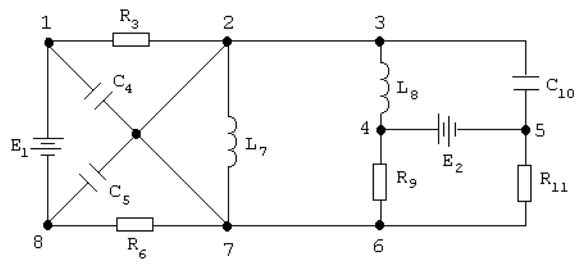

Figure 3

The numbers of the nodes of the circuit are 1,2,3,4,5,6,7,8,9. $\{i, j\}$ are the branches of one. The values in the interior parentheses are the elements of the branches.

Input data.

The circuit :

$\{\{\ \{\{1,2\},\{R_3\}\},\ \{\{1,9\},\{C_4\}\},\ \{\{1,8\},\{E_1\}\},\ \{\{2,9\},\{\}\},\ \{\{2,7\},\{L_7\}\},$
$\{\{2,3\},\{\}\},\ \{\{7,9\},\{\}\},\ \{\{7,8\},\{R_6\}\},\ \{\{8,9\},\{C_5\}\},\ \{3,4\},\{L_8\},$
$\{\{3,5\},\{C_{10}\}\},\ \{\{4,5\},\{E_2\}\},\ \{\{4,6\},\{R_9\}\},\ \{\{5,6\},\{R_{11}\}\},\ \{\{6,7\},\{\}\}\ \}\}$

The order of the graph girth :

"+" - direct (in order of increasing of the numbers of graph vertices), otherwise "-".

Program output.

The Lagrange's function :

$$\frac{L_8(\dot q_1-\dot q_3-\dot q_4+\dot q_5+\dot q_6)^2}{2}+\frac{L_7(\dot q_3-\dot q_7)^2}{2}-\frac{q_4^2}{2C_{10}}-\frac{(q_2-q_5-q_6)^2}{2C_4}-\frac{q_6^2}{2C_5}$$
$$+E_2(q_1-q_4+q_5+q_6)+E_1(-q_1+q_7)\ .$$

The Rayleigh's function :

$$\frac{R_9\dot q_3^2}{2}+\frac{R_{11}(\dot q_1+\dot q_5+\dot q_6)^2}{2}+\frac{R_6(\dot q_1+\dot q_6-\dot q_7)^2}{2}+\frac{R_3(\dot q_1-\dot q_2+\dot q_5+\dot q_6-\dot q_7)^2}{2}\ .$$

The differential equations of the circuit in Lagrange's form :

1. $L_8\ddot q_1-L_8\ddot q_3-L_8\ddot q_4+L_8\ddot q_5+L_8\ddot q_6+(R_{11}+R_3+R_6)\dot q_1-R_3\dot q_2+(R_{11}+R_3)\dot q_5$
$$+(R_{11}+R_3+R_6)\dot q_6+(-R_3-R_6)\dot q_7+E_1-E_2=0\ ;$$
2. $-R_3\dot q_1+R_3\dot q_2-R_3\dot q_5-R_3\dot q_6+R_3\dot q_7+\dfrac{q_2-q_5-q_6}{C_4}=0\ ;$
3. $-L_8\ddot q_1+(L_7+L_8)\ddot q_3+L_8\ddot q_4-L_8\ddot q_5-L_8\ddot q_6-L_7\ddot q_7+R_9\dot q_3=0\ ;$
4. $-L_8\ddot q_1+L_8\ddot q_3+L_8\ddot q_4-L_8\ddot q_5-L_8\ddot q_6+\dfrac{q_4}{C_{10}}+E_2=0\ ;$

5. $L_8\ddot{q}_1 - L_8\ddot{q}_3 - L_8\ddot{q}_4 + L_8\ddot{q}_5 + L_8\ddot{q}_6 + (R_{11} + R_3)\dot{q}_1 - R_3\dot{q}_2 + (R_{11} + R_3)\dot{q}_5$

$$+(R_{11} + R_3)\dot{q}_6 - R_3\dot{q}_7 + \frac{-q_2 + q_5 + q_6}{C_4} - E_2 = 0 \ ;$$

6. $L_8\ddot{q}_1 - L_8\ddot{q}_3 - L_8\ddot{q}_4 + L_8\ddot{q}_5 + L_8\ddot{q}_6 + (R_{11} + R_3 + R_6)\dot{q}_1 - R_3\dot{q}_2 + (R_{11} + R_3)\dot{q}_5$

$$+(R_{11} + R_3 + R_6)\dot{q}_6 + (-R_3 - R_6)\dot{q}_7 + \frac{q_6}{C_4} + \frac{-q_2 + q_5 + q_6}{C_4} - E_2 = 0 \ ;$$

7. $- L_7\ddot{q}_3 + L_7\ddot{q}_7 + (-R_3 - R_6)\dot{q}_1 + R_3\dot{q}_2 - R_3\dot{q}_5 + (-R_3 - R_6)\dot{q}_6$

$$+(R_3 + R_6)\dot{q}_7 - E_1 = 0 \ .$$

3.3. Investigation of stability depending on structure of forces. Let us consider the stability task for trivial solution of differential equation

$$M\ddot{x} \ + (2G + D)\dot{x} \ + (K + P)x \ = Q(x, \dot{x}) \ , \tag{3.6}$$

where $M = M^T > 0$, $D = D^T$, $G = -G^T$, $K = K^T$, $P = -P^T$ are $(n \times n)$ matrices of linear dissipative, gyroscopic, potential and nonconservative forces; $Q(x, \dot{x})$ is single-column matrix of nonlinear forces, $Q(0, 0) = 0$.

This presentation is possible for differential equations of perturbed motion in the neighborhood of steady motion or invariant manifolds of stationary motions. Algorithm to reduce equations to form (3.6) and algorithm to classify nonlinear forces $Q(x, \dot{x})$ suitable for implementation with CAS are described in [8].

Let us define the function V :

$$V = y^T\mathcal{N}y + aF(y) = \dot{x}^T N\dot{x} + x^T Lx + x^T B^T\dot{x} + \dot{x}^T Bx + aF(x, \dot{x}) \ , \tag{3.7}$$

here

$$L = L^T \ , \quad N = N^T \ , \quad \mathcal{N} = \mathcal{N}^T = \begin{pmatrix} L & B \\ B^T & N \end{pmatrix} \ , \quad y = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \ ,$$

a is constant that can be equal to 0 or 1; $F(y)$ is function of power bigger than two.

Let us calculate derivative of function (3.7) for equation (3.6) :

$$\dot{V} = y^T(A^T\mathcal{N} \ + \ \mathcal{N}A)y \ + \tilde{Q}^T M^{-1}\mathcal{N}y + y^T\mathcal{N}M^{-1}\tilde{Q} \ + a\frac{\partial F}{\partial y}(Ay \ + \ M^{-1}\tilde{Q}) \ .$$

Here

$$A = \begin{pmatrix} 0 & E \\ -M^{-1}(K + P) & -M^{-1}(D + 2G) \end{pmatrix} , \quad \tilde{Q} = \begin{pmatrix} 0 \\ Q \end{pmatrix} \ .$$

We require that this derivative was presented in the form :

$$\dot{V} \ = \ \alpha V \ + \ y^T W y \ + \Phi(y) \ .$$

Here $\Phi(y)$ is nonlinear function of power bigger than two. Then we obtain the equation system:

$$A^T\mathcal{N} + \mathcal{N}A - \alpha\mathcal{N} = W \; ; \tag{3.8}$$

$$Q^T M^{-1}(N\dot{x} + Bx) + (\dot{x}^T N + x^T B^T)M^{-1}Q +$$

$$+ \; a \; (\frac{\partial F}{\partial x}\dot{x} - \frac{\partial F}{\partial \dot{x}}M^{-1}((D+2G)\dot{x} + (K+P)x - Q)) = a\alpha F + \Phi(x,\dot{x}) \; .$$

From last equation we find $\Phi(x,\dot{x})$. For eqs. (3.8) we will consider several possibilities.

Variant 1. W is given definitively positive (negative) matrix. From classical theorems it is known that if at least one of characteristic equation roots has positive real part, for equation (3.8) there always exist such $\alpha > 0$ and $\mathcal{N}$ that $y^T\mathcal{N}y$ isn't sign-definite with sign, opposite to sign of W . Vice versa, if for equation (3.8) with $W > 0$ are found $\alpha > 0$ and $\mathcal{N}$ and $y^T\mathcal{N}y$ is not definitively negative, then trivial solution of equation (3.6) is not stable and some roots λ of characteristic equation have positive real part.

Variant 2. $\alpha = 0$, W is given definitively negative (positive) matrix. It is known that if all roots of characteristic equation of system (3.6) have negative real parts, then in (3.8) there is only one solution for $\mathcal{N}$. In these conditions obtained matrix is definitively positive (negative). Vice versa, if with given $W < 0 \; (> 0)$ it's found solution $\mathcal{N}$ of equation (3.8) and $y^T\mathcal{N}y > 0 \; (< 0)$, then zero solution of equation (3.6) is asymptotically stable and all λ of characteristic equation have negative real parts.

If roots λ of characteristic equation are such that $\sum m_i\lambda_i \neq 0$ ($m_i > 0$ are integer, $\sum m_i = 2$, $i = 1,\ldots,n$) and some of them have positive real part, then equation (3.8) have single solution $\mathcal{N}$ and $y^T\mathcal{N}y$ is not definitively positive (negative). Zero solution of system (3.6) is unstable.

Variant 3. $W = 0$. Then equation (3.8) is equivalent to equation [15] :

$$\mathcal{C}z = 0 \; , \tag{3.9}$$

where

$$z = \begin{pmatrix} \mathcal{N}_{1*}^T \\ . \\ . \\ . \\ \mathcal{N}_{2n*}^T \end{pmatrix} , \qquad \mathcal{C} = (A - \alpha E) \otimes E + E \otimes A \; .$$

This equation has nontrivial solution $\Longleftrightarrow \alpha = \lambda_r + \lambda_s$, where $\lambda_r, \; \lambda_s$ are eigenvalues of matrix A .

Remark. *If eigenvalues of matrix A are known and none of them have real part equal to zero, then stability problem can be solved without using Lyapunov functions.*

Variant 4. $\alpha = 0$, $W = 0$. Then we obtain two groups of conditions when equation (3.6) has first integral of form (3.7) :

a)

$$B + B^T - (D-2G)M^{-1}N - NM^{-1}(D+2G) = 0 ; \qquad (3.10)$$

$$(K-P)M^{-1}N - NM^{-1}(K+P) + B^TM^{-1}(D+2G) - (D-2G)M^{-1}B = 0 ; \qquad (3.11)$$

$$(K-P)M^{-1}B + B^TM^{-1}(K+P) = 0 ; \qquad (3.12)$$

$$2L = (K-P)M^{-1}N + NM^{-1}(K+P) + B^TM^{-1}(D+2G) + (D-2G)M^{-1}B .$$

b)

$$\dot{x}^T(\frac{\partial F}{\partial x} + 2NM^{-1}Q - (D-2G)M^{-1}\frac{\partial F}{\partial \dot{x}}) \equiv 0 ;$$

$$x^T(2B^TM^{-1}Q - (K-P)M^{-1}\frac{\partial F}{\partial \dot{x}}) \equiv 0 ; \qquad Q^TM^{-1}\frac{\partial F}{\partial \dot{x}} \equiv 0 .$$

Equations of group (a) give necessary and sufficient conditions of first integral existence for linear system (3.6). Equations of group (b) are used to construct $F(x,\dot{x})$ if it exists or to find such Q that eq. (3.7) is an integral. Function $F(x,\dot{x})$ must fulfill the conditions:

$$\frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_j} = \frac{\partial^2 F}{\partial \dot{x}_j \partial \dot{x}_i} , \qquad \frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i} , \qquad \frac{\partial^2 F}{\partial \dot{x} \partial x} = \frac{\partial^2 F}{\partial x \partial \dot{x}} , \quad (i,j = 1, \dots, n).$$

System (a) (3.10)–(3.12) always has trivial solution $N = 0, \ B = 0$ ($B^T = 0$). It is a system of uniform linear equations and can have (i) only trivial solution or (ii) set of solutions. If there exists nontrivial solution N^* , B^* then $N_1^* = N^* \pm N^{*T}$, $B_1^* = B^* \pm B^{*T}$, $N^{**} = \mathcal{P}N^*\mathcal{Q}^T + \mathcal{Q}N^*\mathcal{P}^T$, $B^{**} = \mathcal{P}^TB^*\mathcal{Q} + \mathcal{Q}^TB^*\mathcal{P}$ (where $\mathcal{P}, \ \mathcal{Q} -$ are matrices commutative with $M^{-1}(D+2G)$, $M^{-1}(K+P)$) are solutions too. Existence conditions for nontrivial solution give us limitations on matrices of original system.

Let us outline some properties of matrix equations of special form that can be obtained by transformation of equations (3.10)–(3.12) :

(i) equation $\mathcal{A}X - X\mathcal{A}^T = 0$ ($\mathcal{A}, \ X$ are $n \times n$ matrices) always has symmetric nontrivial solution; skewed-symmetric solution exists if matrix $\mathcal{A}$ has multiple eigenvalues with corresponding prime elementary divisors and/or different Jordan blocks;

(ii) equation $\mathcal{A}X + X\mathcal{A}^T = 0$ has nontrivial solution if some of roots of characteristic polynomial $f(\mu)$ of matrix $\mathcal{A}$ are equal to zero and/or $f(\mu)$ is even function.

Before writing out equations (a) and (b) in scalar form it is feasible to discuss the possibility of solving the system in matrix form.

Let $D = 0$, $P = 0$, $Q = 0$ (system with potential and gyroscopic forces). The system (3.10)-(3.12) has the solutions :

1. $N = M, \ B = 0$ (this solution determine the integral of energy

$V_1 = \dot{x}^T M\dot{x} \ + \ x^T Kx \ = const$) ;
2. $N = K - 4GM^{-1}G$, $B = 2G^T M^{-1} K$ (this solution determine first integral $V_2 = \dot{x}^T K\dot{x} + (Kx + 2G\dot{x})^T M^{-1}(Kx + 2G\dot{x}) = const$) ;
3. $B = G$, $N = 0$, if $K = \alpha M$ (this solution determine first integral

$$V_3 = 2\dot{x}^T Gx \ + \ x^T GM^{-1}Gx = const \ ). \tag{3.13}$$

The calculated first integrals (or their bunches) can be used to construct Lyapunov functions in proving the stability of the trivial solution for the system (3.6) if Sylvester's inequalities of sing-definiteness of the quadratic part of the integral (3.7) with respect to all variables of the problem are satisfied.

From theorems of Lyapunov's second method it follows that if $K > 0$ then trivial solution of linear system (3.6) is stable with respect to variables $\dot{x}, x$. If matrix K is not definitely positive, we generate a bundle of forms V_1 and V_2 (or V_3 (3.13)). Conditions of sign-definitiveness of this band give sufficient conditions of stability for the critical system in question.

If obtained integral is not sign-definite, it is possible to consider a combination $W = \sum V_i^2 \ \ (i = 1, 2, \ldots, p)$ as Lyapunov function. Then it is necessary to test equations $V_i = 0 \ (i = 1, \ldots, p)$ for compatibility. Conditions of incompatibility of these equations will be sufficient conditions of the trivial solutions of system (3.6). Let us consider a gyroscopic system with two degrees of freedom:

$$\ddot{x}_1 + gx_2 + \alpha x_1 = Q_1 \ ; \qquad \ddot{x}_2 - gx_1 + \alpha x_2 = Q_2 \ . \tag{3.14}$$

Linear system allows first integrals ($M = E$) :

$$H = \dot{x}_1^2 + \dot{x}_2^2 + \alpha(x_1^2 + x_2^2) \ = \ h \ ; \tag{3.15}$$

$$V_3 = g^2(x_1^2 + x_2^2) + 2g(\dot{x}_1 x_2 - x_1\dot{x}_2) \ = \ c_1 \ . \tag{3.16}$$

If $\alpha < 0$ then integral H (3.15) is sign-variable; integral V_3 (3.16) is always sign-variable. Let us build the linear bundle: $V = H + \lambda V_3$. Conditions of definite positivity of this form lead to requirement: $\lambda g^2 + \alpha - \lambda^2 g^2 > 0$. The real parameter λ can be found only if $g^2 > -4\alpha$. We obtained the condition of gyroscopic stabilization of unstable linear potential system (3.14).

With influence of nonlinear forces integrals can change and even disappear. Let us found such nonlinear forces that would keep quadratic parts of the integrals. We substitute values of matrices N, B to equations (b).
For first integral we have (let $F(x, \dot{x}) = F(x)$) :

$$\dot{x}^T (\frac{\partial F}{\partial x} + 2Q) \equiv 0 \ . \tag{3.17}$$

For second integral we have:

$$\dot{x}^T(\frac{\partial F}{\partial x} + 2G\frac{\partial F}{\partial \dot{x}}) \equiv 0 \ , \quad x^T(2G^T Q - K\frac{\partial F}{\partial \dot{x}}) \equiv 0 \ , \quad Q^T\frac{\partial F}{\partial \dot{x}} \equiv 0 \ . \tag{3.18}$$

If we choose $Q = f'(x^T x)x$, ($f(x^T x)$ is a scalar function of variable $z = x^T x$, $'$ is a derivative by x), then equations (3.18) are fulfilled for matrices $N,\ B$ of integral V_3 (3.16) when $F(x, \dot{x}) = 0$. It means that integral (3.16) is kept. For matrices $N,\ B$ of integral $H = h$ (3.15) equations (3.17) give the solution $F(x, \dot{x}) = -f(x^T x)$. Consequently, for nonlinear system the energy integral has the form : $H = \dot{x}^T\dot{x} + \alpha x^T x - f(x^T x)$. Condition of gyroscopic stabilization is kept, because with $g^2 > -4\alpha$ we can choose such λ that $V >> 0$.

If $\alpha = 0$ (linear potential forces are absent) and $f(x^T x) > 0$, then V is sign-variable function.

Let us construct Lyapunov function in form : $W = H^2 + V_3^2$. It will be definitively positive if following conditions will not be fulfilled simultaneously:

$$H = 0 \ , \quad V_3 = 0 \ . \tag{3.19}$$

For example, let $f(x^T x) = (x_1^2 + x_2^2)^2$. Then equations (3.19) have the form :

$$x_1^2 + x_2^2 = (x_1^2 + x_2^2)^2 \ ; \quad g^2(x_1^2 + x_2^2) + 2g(\dot{x}_1 x_2 - x_1\dot{x}_2) \ = 0 \ .$$

If $g \neq 0$ they never are fulfilled simultaneously. Thus if $g \neq 0$ trivial solution of discussed nonlinear system will be stable with respect to coordinates and velocities.

3.4. On reducing first integrals to the simplest form. The usage of Routh-Lyapunov's theorem in the process of qualitative investigation of the phase space suggests the possibility of both obtaining the stationary sets of first integrals and obtaining sufficient conditions of their stability. These results aid in solving a number of problems closely bound up with this problems. As an example we can refer to the problems of branching the families of invariant manifolds of steady motions (IMSM), as well as to the problems of transforming differential equations and their fist integrals to the simplest forms. Consider an example of such an investigation.

Consider a system with gyroscopic forces :

$$\ddot{x} = ax + Ap\dot{y} \ ; \quad \ddot{y} = by - Ap\dot{x} \ , \quad \text{when} \ \ a > 0 \ , \ \ b > 0 \ ,$$

which is known to assume two quadratic first integrals

$$2H = \dot{x}^2 + \dot{y}^2 - ax^2 - by^2 = 2h \ ;$$

$$V = 2by\dot{x} - \frac{(b-a)}{2Ap}\dot{x}^2 - (Apb + \frac{(b-a)b}{2Ap})y^2 - 2ax\dot{y} + \frac{(b-a)}{2Ap}\dot{y}^2 - (Apa - \frac{(b-a)a}{2Ap})x^2 \ . \tag{3.20}$$

Note, our integrals undergo decomposition into pairs independent of forms (functions) of two variables.

Let us conduct analysis of the stationary sets of the complete linear bundle (2.1) of our first integrals $K = \lambda H + V$, $\lambda = const$. The stationarity conditions K are as follows :

$$\frac{\partial K}{\partial \dot{x}} = (\lambda - \frac{(b-a)}{Ap})\dot{x} + 2by = 0 \ ; \quad \frac{\partial K}{\partial x} = -2a\dot{y} - (\lambda + 2Ap - \frac{(b-a)}{Ap})ax = 0 \ ;$$

$$\frac{\partial K}{\partial \dot{y}} = (\lambda + \frac{(b-a)}{Ap})\dot{y} - 2ax = 0 \ ; \quad \frac{\partial K}{\partial y} = 2b\dot{x} - (\lambda + 2Ap + \frac{(b-a)}{Ap})bx = 0 \ . \tag{3.21}$$

They obviously have a trivial solution (equilibrium state). Furthermore, this solution allows us to obtain a stationary value of K for any λ. The conditions of sign definiteness for K generate a sufficient stability condition for this solution, which has the form :

$$A^2p^2 - (\sqrt{a} + \sqrt{b})^2 > 0 \ .$$

As can be easily verified, it coincides with the necessary condition with the precision up to the boundary. Besides the trivial solution for

$$\lambda_1 = -Ap + \frac{\sqrt{M}}{Ap} \ , \quad \lambda_2 = -Ap - \frac{\sqrt{M}}{Ap} \ , \quad (\ M = A^4p^4 + (b-a)^2 - 2A^2p^2(b+a) \ )$$

the system (3.21) also has two IMSMs in the capacity of the solutions :

$$\delta = (\lambda_1 - \frac{(b-a)}{Ap})\dot{x} + 2by = 0 \ , \quad \zeta = (\lambda_2 - \frac{(b-a)}{Ap})\dot{x} + 2by = 0 \ ;$$

$$\xi = (\lambda_1 + \frac{(b-a)}{Ap})\dot{y} - 2ax = 0 \ , \quad \eta = (\lambda_2 + \frac{(b-a)}{Ap})\dot{y} - 2ax = 0 \ . \tag{3.22}$$

So, at least two IMSMs of codimension 2 branch from each of the equilibrium positions for $M > 0$. Furthermore, such a bifurcation takes place from all the stable equilibrium positions and from unstable ones for

$$Ap < \sqrt{a} - \sqrt{b} \ , \quad Ap > \sqrt{b} - \sqrt{a} \ , \quad (\ \sqrt{a} > \sqrt{b} \ ) \ .$$

By direct computation it is possible to prove the following statement : *The transition from the variables* $x, \ y, \ \dot{x}, \ \dot{y}$ *to the variables* $\xi, \ \eta, \ \zeta, \ \delta$ *characterizing deviations* *from IMSMs (3.22), simultaneously transforms the first integrals (3.20) to the sum of squares.*

Indeed, in terms of the new variables the initial differential equations have the form:

$$\dot{\xi} = \frac{\sqrt{M} - L^0}{2Ap}\delta\ , \quad \dot{\delta} = -\frac{\sqrt{M} - L}{2Ap}\xi\ , \quad \dot{\eta} = -\frac{\sqrt{M} + L^0}{2Ap}\zeta\ , \quad \dot{\zeta} = \frac{\sqrt{M} + L}{2Ap}\eta\ ,$$

where $L = A^2p^2 + b - a$; $L^0 = A^2p^2 - b + a$. Hence, the system represented in terms of the new variables has underwent decomposition into the two subsystems, each one having one first integral of the form of a sum of squares :

$$V_1 = (\sqrt{M} + L^0)\zeta^2 + (\sqrt{M} + L)\eta^2\ ; \quad V_2 = (\sqrt{M} - L)\xi^2 + (\sqrt{M} - L^0)\delta^2\ . \qquad (3.23)$$

It can be easily verified that the initial first integrals (3.20) in terms of new the variables are expressed in the form of linear combinations of integrals (3.23).

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