Mathematica and didactical innovation

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ABSTRACT. A package for the education in mathematics focuses on the study of a conic is presented. The package is structured in an hypertextual way, including a part of theory and a guided lab for the practical session.The most meaningful innovation is given by the verification routines. The user may solve exercises by hand and check the solutions he/she finds using two verification functions.

1. INTRODUCTION

Since the first time they were introduced, Mathematica and all similar systems of C.A. (Computer Algebra) ,have been useful to change both the way people looked and thought of didactics in mathematics. Students are no longer passive watchers of a process which used to be distant and unrelated to them. Although a teaching methodology essentially meant for a major participation of the student has always been available, using new teaching tools supported by computer software have opened possibilities only superficially explored. The visualisation through computation and testing are now becoming a fundamental condition for learning.

The computer based education (CBE), or as Kulik (1986) used to call it, "the third mile stone" in the development of computers and the related didactics , represents a link between the traditional teaching techniques and computer technology and it also offers to the tutors a unique opportunity to widen their educational horizons. As a matter of fact what results stimulating for C.B.E. users is the desire of finding and employing the "new", to redefine the "progress report" (Mac Daniel l985). Efficient computer based tutors can be realised inside a Notebook of Mathematica software.

The aim of this work is to describe a package for the education in mathematics focused on the study of a conic. The package is structured in an hypertextual way, including a part of theory and a guided lab for the practical session and a prototype has been realised at both Department of Computer Science and Applied Mathematics at the Faculty of Engineering and the CRMPA -Centre for the Research in Pure and Applied Mathematics - situated in the University of Salerno. In the tutorial the conics

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are introduced starting from the definition and proceeding with the classification and the reduction to a canonical form using both the method of invariant and the rotation/translation. Moreover are defined the hyperbola, ellipsis and the parabola as geometrical loci and because of that we can also include in the target secondary school students.

The laboratory part has been given a structure designed in order to make the final user's interface as friendly as possible. Such an improvement has been obtained by the insertion of buttons and links that make it possible to use all implemented functions without prior knowledge of their syntax. The latter functions allow the user to choose among randomised exercises, whose step-by-step automatic solution (classification, canonical form reduction and graph) may be visualised. For the sake of completeness, besides the affine classification of conics, also the homogeneous counterpart has been considered. The most meaningful innovation is, nonetheless, given by the verification routines. The user may solve exercises by hand and check the solutions he/she finds using two verification functions. The first is entirely automatic: the user enters, as a string, the class (degenerate or not), type(parabola) and shape (with or without centre) of the conic. The routine detects mistakes (using logical inferences), and suggests possible theoretical deepening. Moreover , a manual verification routine has been implemented, allowing to visually verify the values of the parameters that characterise the conic. Again, for the sake of completeness, a subsidiary routine has been implemented that takes as an input of the polynomial associated with the conic, thus allowing the use of the previously described functions.

We stress the fact that the package gives an essentially self-contained, complete environment. In fact, every fundamental step in the study of the chosen subject is covered, form theory to various kinds of exercises. In particular, some diagnostic functionalities are implemented that allow the student to make exercises, have them automatically corrected and even receive suggestions on how to recover from errors.

2. State of the art

Analysing the schedules of some projects published on conference proceedings, we can find out that at the moment different strategies of implementation of mathematical/scientific teaching packages are available through the use of MathematicaTM. One example is the "Transitional Mathematics Project" [1] [2], developed by the Imperial College in London, which basically consists in a pre-course on the use of Mathematics and than a course on a few mathematics algorithm, another example is the project "Rethinking the way we teach undergraduate physics and engineering with Mathematica" [4]. Those project make the most of the front end of Mathematica 2.2.3, trying to point the attention on the schedules contents. Other projects basically utilise the potentiality of the engine of Mathematica for every symbolic numeric and graphic computations, sending the results to the main environment as icons. Example of this strategy is Hypermedia Mathematical Learning Environment (HMLE) [3], developed by the Department of Mathematics of Tampere University of Technology, which consist on an interface developed in C/C++ connected to Mathematica via MathLink. Other example of the same strategy is the development of a prototype by the University of Leeds trough the use of Toolbook to simplify the user interface.

3. Experiments: the lab exercise

The package on the conics has been used for supporting the learning process of first year Engineering students' on this subject. Lectures have been given introducing the theoretical concepts: what a conic is, conics' classification, the standard form of a conic, how to classify and reduce to standard form, and some examples. Then the students have been split into small groups (20 students for each group) and a one-hour exercise was held weekly in a computer classroom. During the computer classroom the students have used the package on the study of a conic. They had two possibility for choosing exercise: a) choose a randomly generated conic; b) give a conic he/she took from a book or he/she invented. In both cases they can choose between a verification function or automatic solution.

The verification routine has been very useful for students. In fact they can verify their level of learning: the student has to give the routine the classification of the given conic, that is the class (degenerate or not), the type (parabola, hyperbola, ellipsis or some couple of lines) and the shape (central or not); if some of the terms he/she gives is not correct, the verification routine points it out and explains which step of the student's process for classification has to be wrong, so the student can study thoroughly the concerning theory as presented in the package. The routine is actually able to do slightly more than this: in fact, it is able to recognize errors of a (most probably) theoretical character (e.g. logical inconsistencies) and computational errors. Correspondently, a different warning message is generated, suggesting the most likely nature of the error and suitable means of correcting it (within the package's environment). This feature proved particularly useful in saving time during the error correction phase, since students did not have to unduely repeat the whole theoretical background in case of mere computational errors and, conversely, received a timely warning when they needed to get a better understanding of the underlying theory therefore, the previous functionality can be regarded - at least from the experimental points of view - as a kind of support, increasing the success rate for a final examination on the subject.

The standard form has been implemented by two methods: the invariants method and rotation-translation method. The student can check each step to have the standard form: in fact the package provides the eigenvalues of the quadratic form associated to the conic, the directions of the new axes individuated by the rotation, and the equations of the rotation and of the translation. Moreover he/she can get the equation of axes and the coordinates of the vertexes and of the centre (if it exists). If the conic is degenerate, the package provides the equation of the lines into which it splits. Finally, the design of the real part of the conic, obtained using the Plot function of *Mathematica*, is given.

Two others facilities are provided: the study of the conic with homogeneous coordinates and a theory notebook introducing the nondegenerate conics as loci.

The first is useful for students which have already studied projective space: the routine provides the homogeneous equation associated to the given conic and the coordinates of the points at infinity and the double points (if any).

The latter facility is interesting for secondary school teachers, because it offers a way to introduce the conics starting from the vantage point of geometrical loci. Then they can give some notions about a more general notion of conic. The package has been used for high school teachers and they have been very excited because they are aware that programs like *Mathematica* offer new way of effective teaching and of making classical subject of some interest for students which are more involved in the learning process as an active part of the learning process itself.

4. Examples

We explore some examples of exercise modules implemented with Mathematica 3.0.

First of all, the package offers to the students the possibility to classify a given conic or a randomly generated one. Students can also check the solving procedure step by step.

Conics classification

Get the package | Available functions | Help

Classify[ ] | Propose | Help | Theory

```
Classify[x^2 + 2 x y + y^2 + x]
```

Automatic solution | Verification | Homogeneous Coordinates omogenee

Invariants method **Invariants**

Rotation - Translation **StandardForm[ ]**

Classify[ ] | Propose | Help | Theory

```
StandardForm[]
```

The rotation needed to have the standard form of the equation of the conic is:

$$x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}$$

$$y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$$

The translation needed to have the standard form of the equation of the conic is:

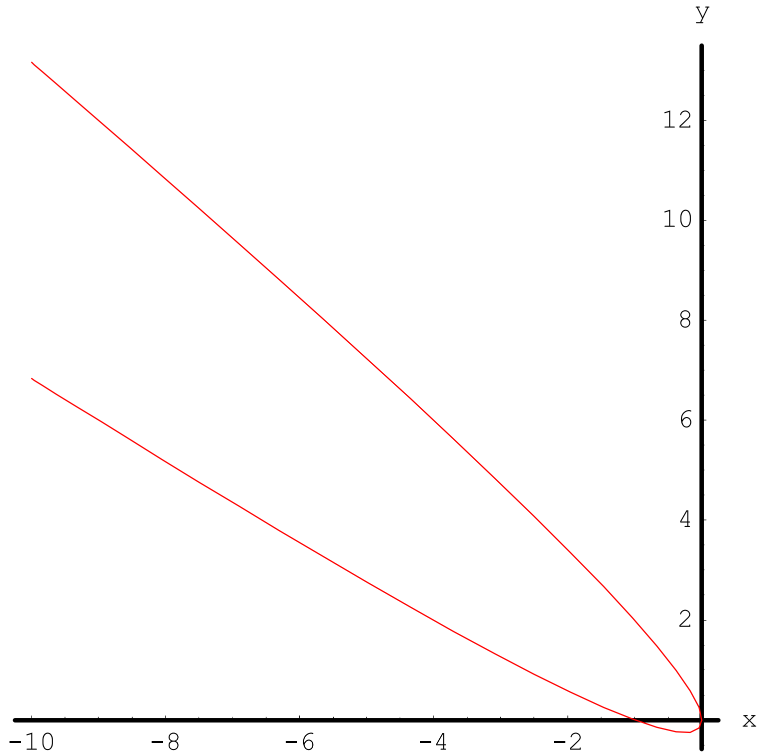
$$x = \frac{1}{4\sqrt{2}} + X$$

$$y = -1 + Y$$

The standard form of the equation is:

$$\frac{1}{2}\left(\frac{1}{16}\left(-1 - 8\sqrt{2}\right) + 2x^2 - \frac{y}{\sqrt{2}}\right) = 0$$

Therefore, the conic is a parabola



Classify[ ] Propose Help Theory

```
Verification["nondegenerate", "parabola", "central"]
```

The conic is nondegenerate, so the computation of the complete matrix determinant is right.

The given conic is a parabola.

Anyway, a parabola is not central.

The determinant of the quadratic form is zero.

It would be better to re-study the theory.

Classify[ ] Propose Help Theory

Classify[$x^2 - 4\,x\,y - 2\,y^2 - 4\,x - 4\,y + 4$]

Automatic solution Verification Homogeneous Co ordinates omogenee

Invariants method **Invariants**

Rotation - Translation **StandardForm[ ]**

```
StandardForm[]
```

The rotation needed to have the standard form of the equation of the conic is:

$$x = \frac{X}{\sqrt{5}} - \frac{2Y}{\sqrt{5}}$$

$$y = \frac{2X}{\sqrt{5}} + \frac{Y}{\sqrt{5}}$$

The translation needed to have the standard form of the equation of the conic is:

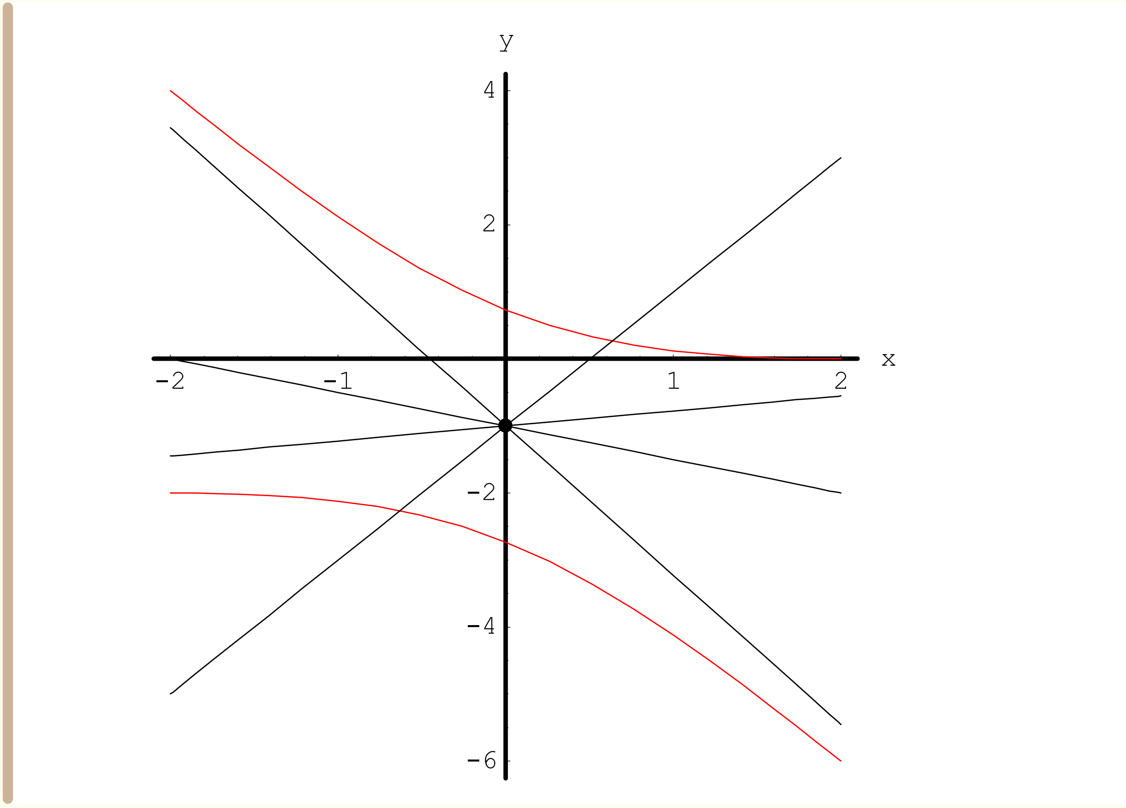
$$x = \frac{2}{\sqrt{5}} + X$$

$$y = \frac{1}{\sqrt{5}} + Y$$

The standard form of the equation is:

$$\frac{1}{6}(6 - 3x^2 + 2y^2) = 0$$

Therefore, the conic is a hyperbola



Classify[ ] **Propose** **Help** **Theory**

```
Verification["degenerate", "hyperbola", "central"]
```

The conic is nondegenerate, so the computation of the complete matrix determinant is wrong.

Anyway, a hyperbola is not central.

It would be better to re-study the theory.

Classify[ ] **Propose** **Help** **Theory**

```
Classify[3 x^2 + 2 x y + 3 y^2 + 2 x - 2 y - 3]
```

A utomatic solution **Verification** **Homogeneous Co ordinates omogenee**

Invariants method **Invariants**

Rotation - Translation **Standard Form[ ]**

The complete matrix determinant is not zero, so the conic is nondegenerate.

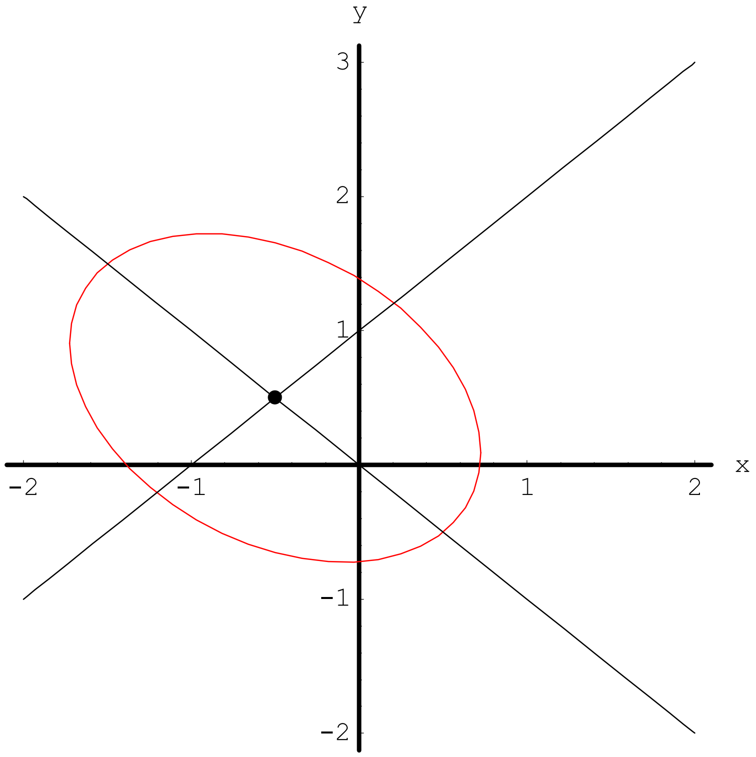
The determinant of the quadratic form is not zero: the conic is central.

In particular, it is greater than zero,

while the complete matrix determinant is less than zero,

in addition eigenvalues are not the same, so it is a real ellipse:

$$\frac{x^2}{2} + y^2 == 1$$



Classify[ ]　Propose　Help　Theory

Invariants Verification **Verification[ ]**

Rotation - Translation Verification **Verification**

The following buttons are useful for a step to step verification

ConicAxes

Rotation　**Traslation**　**Vertexes**　**Eigenvalues**　**Eingenvectors**

The rotation needed to have the standard form of the equation of the conic is:

$$x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}$$

$$y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$$

The translation needed to have the standard form of the equation of the conic is:

$$x = X$$

$$y = -\frac{1}{\sqrt{2}} + Y$$

The eigenvalues are: {2,4}

```
Classify[2 x^2 + 3 x y + y^2 + 4 x + 3 y + 2]
```

Automatic solution **Verification** **Homogeneous Co ordinates omogenee**

Invariants Verification **Verification[ ]**

The verification using the rotation - translation isn't possible because the conic is degenarate

Invariants method **Invariants**

Rotation - Translation **Standard Form[ ]**

The complete matrix determinant is zero, so the conic is degenerate.

The determinant of the quadratic form is not zero: the conic is central.

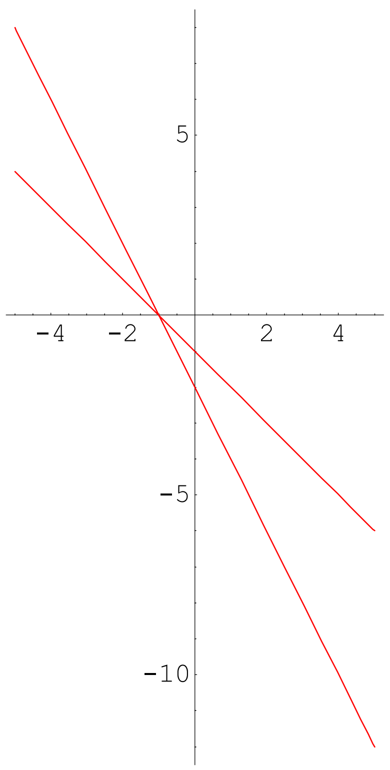
In particular, it is less than zero,

therefore the conic is a pair of intersecting straight real lines, their equations are:

1+x+y = 0

2 (1+x)+y = 0

The co-ordinates of the intersection point are (-1,0).



Classify[ ] Propose Help Theory

```
Verification["degenerate", "intersecting real lines", "central"]
```

Your classification is right.

Classify[ ] Propose Help Theory

```
Classify[x^2 + y^2 + 2 x y - 4 x - 4 y + 4]
```

Automatic solution Verification Homogeneous Coordinates omogenee

The equation of the conic in homogeneous coordinates is :

$(x_1 + x_2 - 2 x_3)^2 = 0$

I punti impropri della conica sono:

(-1, 1, 0)

(-1, 1, 0)

The conic has infinit double points, i.e. all points belonging to the line having the equation : .

$x_1 + x_2 - 2 x_3 = 0$

Therefore the conic is 'twice degenerate'.

Invariants method Invariants

Rotation - Translation StandardForm[ ]

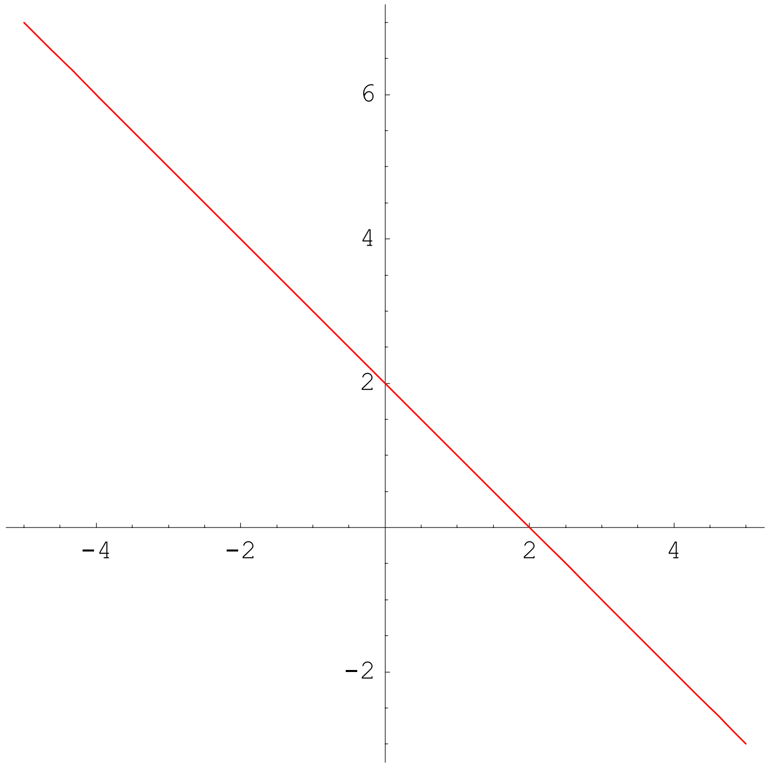
```
StandardForm[]
```

The proposed conic is degenarate. Its components are:

$-2 + x + y = 0$

$-2 + x + y = 0$

The conic is therefore a single straight real line 'counted twice'.



Classify[ ] Propose Help Theory

Classify[$x^2 + 2\,x\,y + y^2 - 2\,x - 2\,y$]

A utomatic solution Verification Homogeneous Co ordinates omogenee

Invariants method Invariants

Rotation - Translation Standard Form[ ]

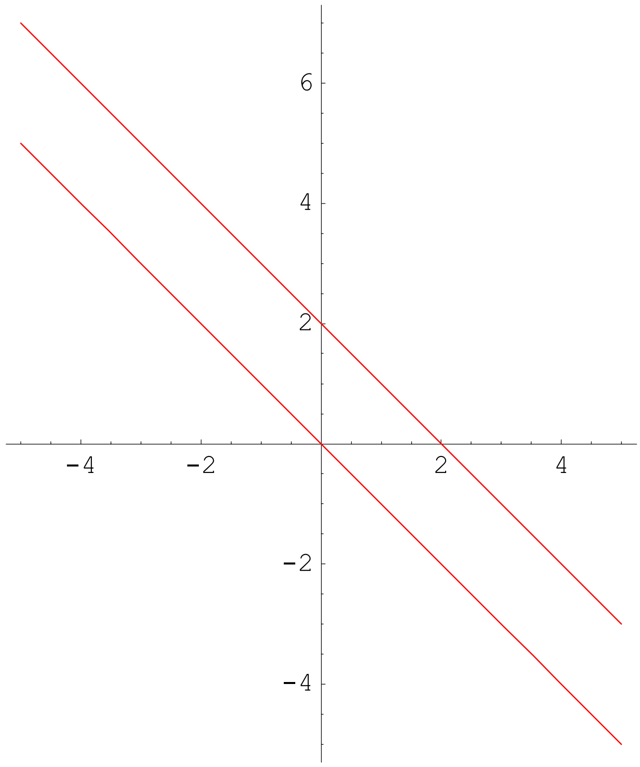
The complete matrix determinant is zero, so the conic is degenerate.

The determinant of the quadratic form is zero

in particular it is a pair of parallel straight real lines

$-2 + x + y = 0$

$x + y = 0$



Classify[ ] Propose Help Theory

Classify[$x^2 + y^2 + 2\,x\,y + 1$]

Automatic solution Verification Homogeneous Co ordinates omogenee

Invariants method **Invariants**

Rotation - Translation **Standard Form[ ]**

The complete matrix determinant is zero, so the conic is degenerate.

The determinant of the quadratic form is zero

in particular it is a pair of parallel straight complex lines

$I + x + y = 0$

$-I + x + y = 0$

Classify[ ] Propose Help Theory

```
StandardForm[]
```

The proposed conic is degenerate. Its components are:

$I + x + y = 0$

$-I + x + y = 0$

Therefore the conic is a pair of parallel straight complex lines.

Classify[ ] **Propose** **Help** **Theory**

$$\text{Classify}\left[2 x^2 + 2 y^2 + 1 + 2 x y + 2 \sqrt{\frac{3}{2}}\, y\right]$$

A utomatic solution **Verification** **Homogeneous Co ordinates omogenee**

Invariants method **Invariants**

Rotation - Translation **Standard Form[ ]**

The complete matrix determinant is zero, so the conic is degenerate.

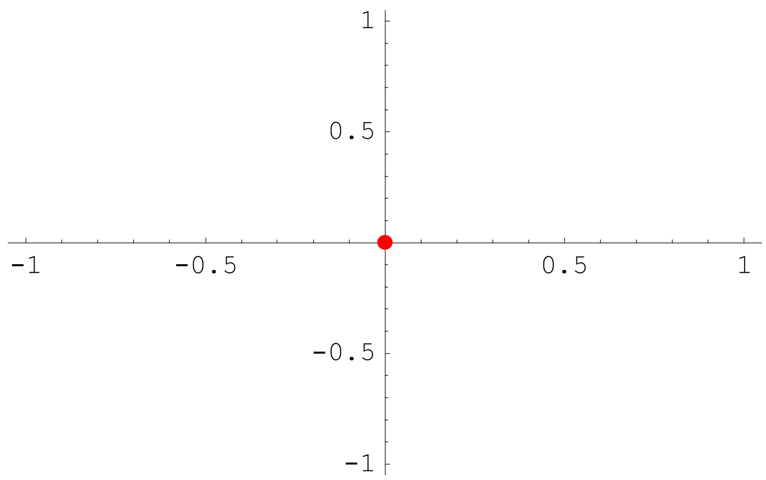
The determinant of the quadratic form is greater than zero

it is a pair of parallel striagt complex lines intersecting in a single real point, its equations are:

$$\frac{1}{4}\left(\sqrt{6} + 2 x + \sqrt{2}\sqrt{-1 + 2\sqrt{6}\, x - 6 x^2}\right) + y = 0$$

$$\frac{1}{4}\left(\sqrt{6} + 2 x - \sqrt{2}\sqrt{-1 + 2\sqrt{6}\, x - 6 x^2}\right) + y = 0$$

In this case it is possible the real conic part can be drawn, that is the lines intersection point, which co-ordinates are (0,0).



Classify[ ] Propose Help Theory

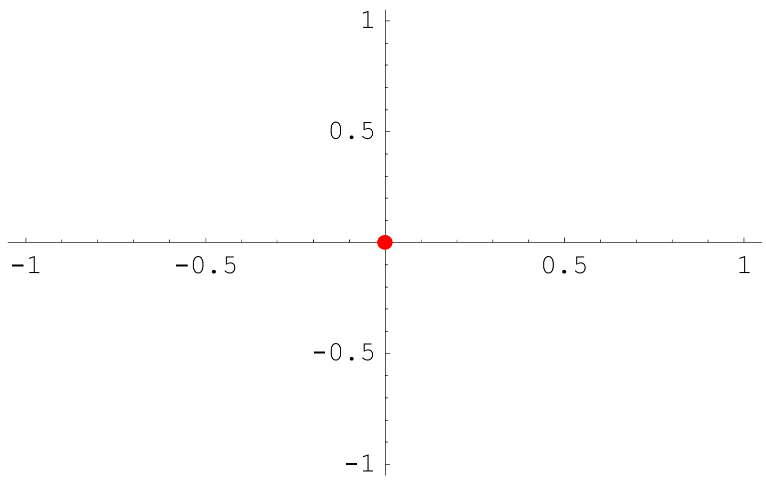
```
StandardForm[]
```

The proposed conic is degenerate. Its components are:

$$\frac{1}{4}\left(\sqrt{6}+2x+\sqrt{2}\sqrt{-1+2\sqrt{6}x-6x^2}\right)+y=0$$

$$\frac{1}{4}\left(\sqrt{6}+2x-\sqrt{2}\sqrt{-1+2\sqrt{6}x-6x^2}\right)+y=0$$

Therefore the conic is a pair of intersecting straight complex lines



Classify[ ] Propose Help Theory

5. Conclusions

The possibilities offered by Mathematica 3.0 are relevant from the educational point of view in order to produce efficient computer based tutors. It allows the realisation of training modules of high cognitive and didactic content. The package presented in this paper represents a user-friendly way to learn some fundamental concepts by oneself.

References

[1] Kent, P. & Ramsden, P. & Wood, J. The Transitional Mathematics Project, The CTI Maths & Stats Newsletter, 1994, 5(1), Mathematics with Vision, Proceedings of the First International Mathematica Symposium.

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