# Testing Chaos and Fractal Properties in Economic Time Series

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## ■ Abstract

Search for empirical evidence of chaos and testing fractal and other statistical properties in the framework of time series analysis are carried on as a preparatory step in order to apply these concepts to data proper of Financial Markets and deal with the puzzling failure of traditional economic theories.
Concepts like correlation dimension and Lyapunov exponents are discussed and simple *Mathematica* programs are given for their evaluation. Before their application to real economic data, a test on a well known nonlinear dynamical system, through the correspondent reconstructed phase space and time series, is carried out.

# ■ 1. Introduction

As well known, economic theories have traditionally been dominated by a linear modelling, based on concepts like Gaussian distributions and random walks, generating the so called Capital Market Theory built on the assumption of normally distributed returns and the Efficient Market Hypothesis (EMH), by which the markets follow a random walk. As a first consequence the future would be unrelated to the past or the present, with no possibility of identifying trends or cycles.
On the contrary, as is proven by several results starting from the discrepancy between the natural consequences of the Efficient Market hypothesis and the real behavior of financial time series, financial markets behave in a non-linear fashion [1, 2, 3, 4]. Actually several empirical studies have attempted to prove the Gaussian assumption, but have often delivered contrary results.
Since the first works by Mandelbrot [5], many efforts have been made to extract suitable information from the behavior of financial markets in order to include their description in the framework of chaos theory and sciences of complexity and non-linearity [1, 2, 6].
Mandelbrot hypothesis and the related fractal analysis, by which returns belong to a family of Stable Paretian distributions, have already been discussed in a previous paper [7] with particular emphasis on the comparison between empirical results and the analysis based on the so-called Fractal Market Hypothesis [1, 3]. In this model, the returns follow a biased random walk, mathematically described by the so-called fractional Brownian motion and empirically obtained through fractal time series. In [7], based on the original ideas by Mandelbrot, the R/S analysis was carried out and the correspondent Hurst exponents evaluated, as a first step in the recognition and characterization of the complex dynamics underlying economic systems and its relation with fractional Brownian motion [1, 8 ]. This analysis, being well suited to distinguish a random series from a non-random one, allows us to give a qualitative description of the behavior of markets.
Fractal time series are characterized as long memory processes. They possess cycles and trends, and are the result of a nonlinear dynamics and deterministic chaos. Information is not immediately reflected in prices, as the EMH states, but instead manifests itself as a bias in returns.
Fractal analysis and chaos theory are closely related, in the sense that nonlinear systems can be statistically described using fractals, and analytically examined using chaos theory.

In the present paper, thanks to the potentiality of *Mathematica* in handling long time series and to the availability of special Packages like "Statistics", an empirical investigation of the properties of discrete time series is carried out, within the framework of non-linear dynamic theory. This is pursued in order to look beyond random walks and related theories, towards models of complexity and

deterministic chaos, in which chaotic properties are generated by the intrinsic nonlinearities of the system.
Through concepts like phase space recostruction, correlation dimension and Lyapunov exponents, properly adapted to time series, we try to give a quantitative measure of correlations and dimensions and investigate the existence of long-term correlations and the fading away of initial information.
The final aim is to apply these programs to the distributions of stock market price changes, but preliminarly they have been tested on a well known non-linear model, for which the existence of deterministic chaos has been separately proved.

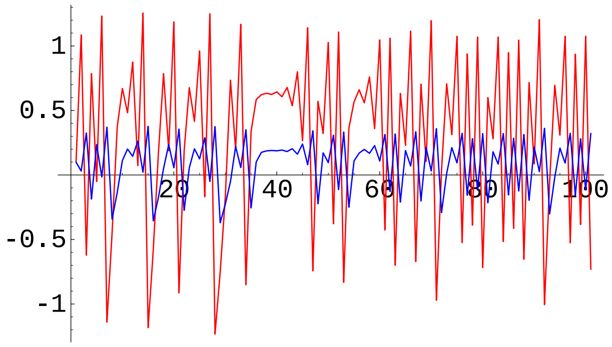
## ■ 2. Time Series Analysis and Correlation Dimension

In dynamical systems theory, chaos means irregular fluctuations in a deterministic system: the system behaves irregularly because of its own internal structure, and not because of random forces acting from outside. Quite a voluminous literature regarding nonlinear dynamics exists (see, for example Refs [9], [10]), but it is mostly based on non-linear differential equations, one for each variable describing the system, all of them defining the so-called phase space. The equations usually depend on a few control parameters, for some values of which such systems show up chaotic behavior.
The new aspects raised by dynamical systems theory are the implied geometric view of temporal behavior and the existence of geometric invariants, such as fractal dimensions and Lyapunov exponents.
The difficulty of studying stock markets data or any other experimental series of data coming from the empirical monitoring of observations, is that, in all these cases, all information at our disposal consists in discrete sequences of numbers, without any reference to differential equations.
This difficulty has been overcome thanks to the idea of phase space reconstruction by Ruelle [11]. This allows the application of dynamical systems techniques to series of data, usually obtained by monitoring the value of a single observable as a function of time, with the consequent extraction of geometric information from it. The general idea is to generate several different scalar signals from the original signal x(t) obtained through time delays.
For example, for the Henon map which is mathematically described in a two-dimensional space obtained by the variables {x(t), y(t)} satisfying the system of non linear equations

```
x[0] = .1;
y[0] = .1;
a = 1.4;
b = 0.3;
x[t_] := x[t] = 1 + y[t - 1] - a * x[t - 1] ^ 2
y[t_] := y[t] = b * x[t - 1]
```

we can extract the following discrete time series of values plotted together:

```
ff = Table[x[t], {t, 0, 100}];
gg = Table[y[t], {t, 0, 100}];
p1 = ListPlot[ff, PlotJoined -> True,
  PlotStyle -> RGBColor[1, 0, 0], DisplayFunction -> Identity];
p2 = ListPlot[gg, PlotJoined -> True,
  PlotStyle -> RGBColor[0, 0, 1], DisplayFunction -> Identity];
Show[p1, p2, DisplayFunction -> $DisplayFunction]
```

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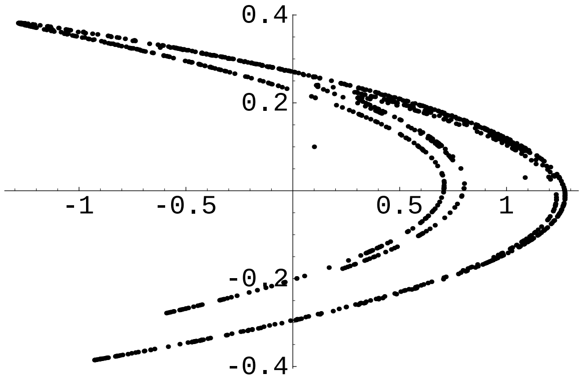
and the graphical representation showing the existence of the Henon attractor:

```
Henon := Table[{x[t], y[t]}, {t, 0, 1000}]
H = ListPlot[Henon]
```

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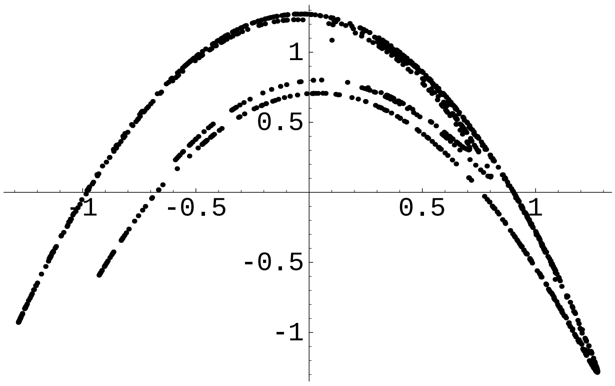
On the other hand,the existence of the attractor can also be obtained through the delayed variables {x(t),x(t+$\tau$)}, for fixed time lag $\tau$ and the corresponding reconstructed two-dimensional phase space portrait.
In this way it is possible to reconstruct a picture topologically equivalent to the Henon map attractor in two-dimensional space by measuring only one of its coordinates, x(t), and plotting the delay coordinate {x(t),x(t+$\tau$ )}.

```
xl[t_] := x[t + 1]
Henonlag := Table[{x[t], xl[t]}, {t, 0, 1000}]
Hlag = ListPlot[Henonlag]
```




```
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There is large empirical evidence that these reconstruction methods preserve the geometry and the geometric invariants of the dynamical systems under investigation [11]. In addition, starting from a single variable series, this technique can be easily trasferred to the analysis of generic discrete time series, like those typical of the financial market.

Moreover, this idea can be extended in order to obtain, starting with a single time series, different lagged time series $x_k(t)=x(t+(k-1)\tau)$, with k=1 ...d, where d is the embedding dimension.

In this way, from one single scalar signal, one can reconstruct the dynamics in a finite d-dimensional space and study d-dimensional orbits obtained by the time-delay method. Search for the optimal embedding dimension is crucial in order to obtain information about the underlying system and is related to the fractal dimension of the phase space itself. Actually attractors usually have fractal dimensions and, due to their typical non linear character, will retain these dimensions as we increase the embedding dimension. The next higher integer to the fractal dimension tells us the minimum number of dynamic variables we need, in order to model the dynamics of the system.

A practical method to estimate the fractal dimension of the phase space goes through the calculation of the correlation dimension as in [12].

This technique is based on the so-called correlation integral, that is an estimate of the probability that two points on the attractor lay less than a distance R from each other. Given the Npt values of the series and for fixed embedding dimension d and time lag $\tau$ , we reconstruct the phase space and calculate the percentage of points within a certain distance R from one another, for increasing values of R, through the correlation integral Corr(R). The following is an example with d=3 and $\tau$=1 for the Henon map:

```
tau := 1;
Npt := 1000
dim := 3
DT = 10;

x2l[t_] := x[t + (dim - 1) * tau]

X = Table[x[t], {t, 0, Npt + DT}];
Y = Table[xl[t], {t, 0, Npt + DT}];
Z = Table[x2l[t], {t, 0, Npt + DT}];
```

```
P1[i_] := P1[i] = X[[i]]
P2[i_] := P2[i] = {X[[i]], Y[[i]]}
P3[i_] := P3[i] = {X[[i]], Y[[i]], Z[[i]]}
```

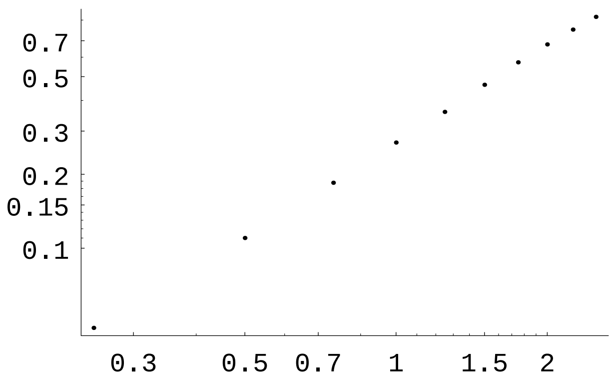
```
Hs[s_] := If[s > 0, 1, 0]     (* Heaviside function *)
dist[P_, Q_] := N[Sqrt[Sum[(P[[k]] - Q[[k]])^2, {k, 1, dim}] ]]
Corr[R_] :=
 N[(Sum[Hs[R - dist[P3[i], P3[j]]], {i, 1, 200}, {j, 1, i - 1}]
  + Sum[Hs[R - dist[P3[i], P3[j]]], {i, 1, 200}, {j, i + 1, 200}]) /
   200^2]
```

```
datacorr = Table[{R, Corr[R]}, {R, 0.25, 2.5, 0.25}]
```

```
{{0.25, 0.04735}, {0.5, 0.11005}, {0.75, 0.18475},
 {1., 0.26945}, {1.25, 0.35915}, {1.5, 0.46315},
 {1.75, 0.57185}, {2., 0.6765}, {2.25, 0.77735},
 {2.5, 0.8759}}
```

A log/log plot of the output and an estimate of the slope of the linear region of this graph gives the correlation dimension D, due to the fact that, by increasing the value of R, Corr [R] should increase as $R^D$, or, after taking the logarithm of both sides, Log[Corr] = D Log [R] + constant. Following Ref. 12 the value of D should eventually converge, by increasing d, to the true value of the fractal dimension of the attractor. In the following the correlation dimension for the Henon map is estimated for d = 3.

```
Needs["Graphics`Graphics`"]
LogLogListPlot[datacorr]
```




```
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```

```
LP = LogLogListPlot[datacorr, DisplayFunction -> Identity];
lp = Table[LP[[1, i, 1]], {i, 1, Length[datacorr]}];
Fit[lp, {1, x}, x] // Simplify
```

```
-0.563648 + 1.28361 x
```

The output value of the correlation dimension for embedding dimension d =3 results to be 1.28361. This value must be compared with the usually accepted fractal dimension 1.26 for the Henon attractor, directly computed using the box-counting method.

For experimental data, like those coming from stock market time series, correlation integrals should be calculated and the previous program run for increasing values of the embedding dimension and fixed time lag. The fractal dimension should eventually converge to its true value.

First checks with this procedure have been carried out on data coming from economic series. In particular we have taken into account the normalized log-returns coming from the daily closing prices of the stock "FIAT" on the period of time from January 1973 to December 1995 and already used in a previous paper [ 7] for the evaluation of Hurst exponents. Using exactly the same definitons and procedures as before, we obtain, for embedding dimension d=2 a correlation dimension equal to 1.41521, and, for embedding dimension d=3, a correlation dimension equal to 1.38841. This can be considered a positive result, taking into account the relation D = 2 - H , that should hold between the fractal dimension D and the Hurst exponent H, and the values, between 0.564 and 0.612, of H, as found in [7].

## ■ 3. Time Series Analysis and Lyapunov Exponents

A large number of important features, typical of nonlinear dynamics, can be extracted from the simple logistic map f(x) = a x (1-x) of the unit interval [0, 1] into itself, where the parameter a ranges over the interval [0, 4] . As well known, even if the equation is very simple, it produces, due to its nonlinearity, a very complex behavior in correspondence with particular values of the parameter a. This equation has been extensively analysed in the literature, due to the fact that not only it is a prototype of a large class of one-dimensional non linear difference equations but it also shares many properties of higher dimensional nonlinear systems. In particular it exhibits fractal and self-similarity properties, proper of nonlinear feedback processes, together with a sensitive dependence on initial conditions, by which initial neighbouring orbits separate exponentially. It is this rapid separation, characteristic of chaotic solutions, which makes it impossible in practice to predict the behavior of a solution far into the future. This property can be quantified through the evaluation of the Lyapunov exponent $\lambda$, which measures the evolution in time of the distance of two nearby points. For each value of the parameter a, this exponent can be obtained directly from the expression of the function f in the equation, through the definition [9] $\lambda(a) = \text{Limit}\left[\frac{1}{N}\sum_{n=0}^{N-1} \text{Log}[|f'(f^n(x_0))|], N \to \infty\right]$ which shows that $\lambda$ is a measure of the exponential separation of the neighbouring orbits averaged over all points of an orbit around an attractor. The Lyapunov exponent may be interpreted in terms of information theory as giving the rate of loss of information about the location of the initial point or in terms of Kolmogorov entropy as measuring the disorder of the system. $\lambda$ can be evaluated analytically in some simple cases, while for the logistic map it can be easily computed numerically, as shown below.

```
Lyapun[X0_, a_, N_] := Module[{exp},
    f[x_] := a * x (1 - x);
    F[x_] := a (1 - 2 x);
    X[n_] := Nest[f, X0, n];
    exp := Sum[Log[Abs[F[X[n]]]], {n, 0, N - 1}] / N; exp]

Plot[Lyapun[0.3, a, 100], {a, 3.4, 4},
  AxesLabel -> {"a", lambda}]
```

A first glance to the previous picture immediately confirms the distinction between regions with $\lambda < 0$, corresponding to the existence of stable cycles and regions with $\lambda > 0$, corresponding to chaotic attractors. Also a much richer and fine structure of the curve can be appreciated.

For the logistic map the Lyapunov exponent could be evaluated directly from the derivative of the function f , and the same is true also in more dimensions, provided the equations of motion are known.

Differently, when we have to analyse experimental data, like those obtained monitoring a scalar signal for a finite time or a series coming from the stock market, the calculation of the characteristic exponents needs some preliminary steps in order to reconstruct the dynamics in a suitable space. These steps are analogous to those discussed before, regarding the embedding procedure and time delay method.

First of all, if the data come from experimental time series, the entire spectrum of Lyapunov exponents cannot be evaluated. In this case, actually, the so-called trajectory tracing method has been developed [13] in order to calculate at least the largest exponent. Positive values for this exponent would prove the divergence of nearby points in the reconstructed phase space and the existence of a strange attractor with sensitive dependence on initial conditions.

Assume we have a large collection of Npt experimental points from a long time series. The first step is to fix some input parameters, like the maximum length scale $L_{\max}$ and the minimum length scale $L_{\min}$ (corresponding to the estimate of the length on which the local structure of the attractor is no longer to be probed or on which noise is expected to appear, respectively), a costant propagation time DT and a maximum angular error $\theta$ to be accepted at each step. These parameters must be added to the previously defined embedding dimension and time lag, used for the phase space reconstruction.

The calculation begins with the search of the nearest neighbor to the first point

(the fiducial point), omitting points closer than $L_{\min}$. The main loop carries out repeated cycles of propagating and replacing points.
The current pair of points is propagated for a time DT and their final distance evaluated. The log of the ratio of final to initial separation of the pair of points gives a measure of the current divergence of distances. A replacement step is then attempted: the distance of each point from the evolved fiducial point is evaluated and points closer than $L_{\max}$ and further away than $L_{\min}$ are taken into account and the point corresponding to the smallest angular change is used for replacement.
This process is repeated until the fiducial trajectory reaches the end of the data file, when the Lyapunov exponent, being updated at each step, should reach a stationary value.
The idea included in this algorithm consists in following step by step the evolution of a pair of points, the fiducial one and the candidate: each time the distance between these two points becomes too long, a replacement procedure of the candidate is applied in such a way that the orientation between the new pair of points is as close as possible to that of the original pair.
A replacement point is necessary in order to measure stretching but not folding on the attractor. Moreover, this method requires a substantial quantity of data to allow the updating procedure and a sufficiently long evolution time to reach convergence in the evaluation of the exponent.
In the following the previously described algorithm has been tested in order to find the largest Lyapunov exponent for the Henon map.

```
x[0] = .1;
y[0] = .1;
a = 1.4;
b = 0.3;
x[t_] := x[t] = 1 + y[t - 1] - a * x[t - 1] ^ 2
y[t_] := y[t] = b * x[t - 1]

tau := 1;
Npt := 1000
dim := 3
DT = 10;
x1[t_] := x[t + 1]
x21[t_] := x[t + (dim - 1) * tau]

X = Table[x[t], {t, 0, Npt + DT}];
Y = Table[x1[t], {t, 0, Npt + DT}];
Z = Table[x21[t], {t, 0, Npt + DT}];

Lmin = 0.01;
Lmax = 0.1;
theta = 0.3;

P0 = {X[[1]], Y[[1]], Z[[1]]};
P[i_] := {X[[i]], Y[[i]], Z[[i]]}

dist0 = Table[dist[P0, P[i]], {i, 1, Npt}];
d0 = Min[Select[Drop[dist0, 10], # > Lmin &]];
i0 = Flatten[Position[dist0, d0], 1]
```

```
Pt1[k_] := Pt1[k] = P[1 + k * DT] (* fiducial point *)
kmax = Floor[(Npt - 1) / DT];

P0' = Flatten[P[i0], 1];    (* nearest neighbour  *)
P1 := Flatten[P[i0 + DT], 1] (* evolved nearest neighbour *)
d1' = dist[Pt1[1], P1];
w1 := Log[2, d1' / d0]

ang[P1_, P2_, P_] := ArcCos[Abs[
  Sum[(P1[[j]] - P[[j]]) * (P1[[j]] - P2[[j]]),
  {j, 1, dim}]] / (dist[P1, P] * dist[P1, P2])]

DIST[k_] := DIST[k] =
     Join[Table[dist[Pt1[k], P[i]], {i, 1, k * DT - 9}],
     Table[dist[Pt1[k], P[i]], {i, 10 + k * DT, Npt}]];

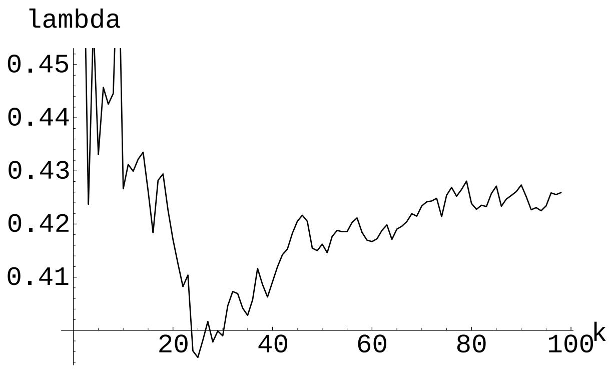
Lyapunov[kmax_] := Module[{k, index, sum, lyap},
k = O;
index = i0[[1]];
sum = 0;
For[k = 1, k < kmax, k++,
DISTANCES = DIST[k];
ANGLES = Chop[Join[Table[ang[Pt1[k], P[index + DT], P[i]],
        {i, 1, k * DT - 9}],
        Table[ang[Pt1[k], P[index + DT], P[i]],
        {i, 10 + k * DT, Npt}]], 10^(-6)];
SELECTION = Select[Select[
     Table[{DISTANCES[[i]], ANGLES[[i]]},
     {i, 1, Length[DISTANCES]}],
     #[[2]] < 5 * theta&], Lmin < #[[1]] < Lmax&];
L = Length[SELECTION];
phi = Table[SELECTION[[i, 2]], {i, 1, L}];
angmin = Min[phi];
d = SELECTION[[Flatten[Position[phi, angmin], 1], 1]][[1]];
ds = Table[dist[Pt1[k], P[i]], {i, 1, Npt}];
index = Flatten[Position[ds, d], 1][[1]];
d' = dist[Pt1[k + 1], P[index + DT]];
w = Log[2, d' / d];
sum = sum + w;
Lyap = (w1 + sum) / ((k + 1) DT);
Print[Lyap]]]
```

The output values of Lyapunov[kmax], after reporting them in the file "lya", have been plotted together with respect to the time evolution index k, in order to show up the asymptotic behavior of the largest Lyapunov exponent for the time series under investigation.

```
lya
```

```
{0.503968, 0.476811, 0.423726, 0.457607, 0.433081, 0.445704,
 0.442564, 0.444572, 0.474226, 0.426638, 0.431215, 0.429938,
 0.432204, 0.433511, 0.426223, 0.418379, 0.428222, 0.429437,
 0.422616, 0.417088, 0.412538, 0.40823, 0.410393, 0.396123,
 0.394902, 0.398119, 0.401653, 0.397807, 0.399868, 0.39896,
 0.40459, 0.407297, 0.406924, 0.404192, 0.402813, 0.405722,
 0.411644, 0.408607, 0.40627, 0.40911, 0.411946, 0.414233,
 0.415292, 0.418268, 0.420522, 0.421656, 0.420505, 0.415461,
 0.414997, 0.416213, 0.414609, 0.41768, 0.418796, 0.418563,
 0.418582, 0.420303, 0.421147, 0.418428, 0.416961, 0.416698,
 0.417217, 0.4188, 0.419838, 0.417105, 0.419046, 0.419588,
 0.420456, 0.421947, 0.421489, 0.423401, 0.424178, 0.424338,
 0.424844, 0.421394, 0.425432, 0.426888, 0.425235, 0.426508,
 0.428075, 0.423871, 0.422765, 0.423541, 0.423288, 0.425749,
 0.42713, 0.423345, 0.424694, 0.425376, 0.426104, 0.42735,
 0.425163, 0.422685, 0.423094, 0.422498, 0.423422,
 0.425861, 0.425541, 0.425924}
```

```
ListPlot[lya,
 PlotJoined -> True, AxesLabel -> {"k", "lambda"}]
```




```
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```

From the previous calculations an asymptotic value for $\lambda = 0.426$ can be extracted, which should be compared with the value 0.40 of the positive Lyapunov exponent for the Henon map obtained directly from the equations of motion. Knowledge of the largest Lyapunov exponent allows us to obtain an idea on how reliable our forecasts are and for what time span: for example a value of 0.426 means that we loose 0.426 bits of predictive power with each iteration.

Of course, the previously described algorithm as in [13] would be exact only for an infinite amount of noise-free data. On the contrary, we have typically to deal with a limited amount of noisy data, coming, for example, from experimental observations. In this case the algorithm can only give an estimate of the largest Lyapunov exponent, and for its calculation parameters like the embedding dimension, the time lag and other input parameters must be chosen with care. The previously described results obtained by using *Mathematica* programs seem to be very encouraging and induce us to do additional checks on specific economic time series.

## ■ 4. Conclusions and Outlook

An analysis of scaling properties of suitably defined correlation and structure functions has been carried out in order to test the existence of anomalous scaling and to control the trend towards Gaussian behavior in the framework of the analysis of discrete time series.
This analysis, based on the study of correlation properties, can be considered as a first contribution in verifying the possibility of the definition of a multiaffine/multifractal stochastic model as the natural framework to describe anomalous power-law scaling of economic time series [14]. The multiaffine description, implying the existence of a hierarchy of multiple time scales and the superposition of different stochastic contributions, each fluctuating at a different time scale, could help in understanding the analogies and differences between turbulence and financial markets [15], in order to obtain a coherent representation of this complex dynamical system. Additional work is of course necessary in order to quantify the effort of classifying economic time series as fractal or chaotic time series and this will be the subject of future work, with particular emphasis on the recognition of trends and nonperiodic cycles.

## ■ References

[1] E. Peters, Fractal Market Analysis, John Wiley and Sons (1994)
[2] E. Peters, Chaos and Order in the Capital Markets, John Wiley and Sons (1996)
[3] B.B. Mandelbrot , Fractals and Scaling in Finance, Springer (1997)
[4] J.P. Bouchaud, P.Cizeau, L.Laloux and M.Potters, Physics World, issue Jan. 1999
[5] B.B. Mandelbrot, The Journal of Business, vol. 36 (1963) 394
[6] P.Cheng, System Dynamics Review, vol.4 (1988) 81
[7] M. Guiducci and M.I. Loffredo, Time Series Analysis: Mandelbrot Theory at Work in Economics, Proceedings IMS'97
[8] R. Maeder, The Mathematica Journal, 6 / 1 (1995) 38
[9] T. Mullin, The Nature of Chaos, Clarendon Press (1994)
[10] H. Abarbanel, R.Brown, J.Sidorowich and L.Tsimring, Reviews of Modern Physics, vol.65 (1993), 1331
[11] D. Ruelle, Chaotic Evolution and Strange Attractors, Cambridge U.P. (1989)
[12] P. Grassberger and I.Procaccia, Phys. Rev. Lett., vol. 52 (1984) 2241
[13] A.Wolf, J.Swift, H.Swinney and J.Vastano, Physica, vol.16D (1985) 285
[14] B.B. Mandelbrot, Multifractals and 1 / f Noise, Springer (1999)
[15] R. N. Mantegna and H.E. Stanley, Nature vol. 383 (1996) 587