Experiments in the Theory of Surfaces

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Abstract

The purpose of this article is to show the advantage of using *Mathematica* in the theory of surfaces.

The examples in the classical differential geometry, namely the theory of curves and surfaces, have been confined to a small group of calculable objects, because of the difficulty in evaluating geometrical quantities and solving differential equations explicitly. But *Mathematica* has made it possible to deal with wide range of objects and to perform experimental treatment of them, based on the power of numerical calculation, the **`NDSolve`** and **`NIntegrate`** commands in particular, and graphical visualization by the **`ParametricPlot`** command. I would like to introduce here several animations and figures that I use in my differential geometry class for undergraduate students. First of all, these graphics help students understand the basic notions of differential geometry. Secondly, we can experiment on geometry through these graphics.

This article is one of the serial talks given by the author at Developers Conference 95, IMS 97 in Rovaniemi, and WMC 98 in Chicago. They are all targeted for the experimental usage of *Mathematica* on differential geometry. This time the topics are focused on the surfaces in the 3-dimensional Euclidean space.

The commands are contained in six notebooks. If the commands in other notebooks cause any trouble, please restart *Mathematica*. See also the explanation of the notebooks.

1. Decomposition of the curvature vector.

Let $c(t)$ be a curve on a surface $f(u, v)$, $\mathfrak{e}_2(t)$ be its unit binormal vector, and (t) be the curvature. Then, the vector $\mathfrak{k}(t)$ $(t)\,\mathfrak{e}_2(t)$, called the curvature vector, is decomposed into the direct sum $\mathfrak{k}(t)$ $\mathfrak{k}_n(t)$ $\mathfrak{k}_g(t)$ of the normal part (normal curvature vector) and the tangential part (geodesic curvature vector). If two curves on $f(u, v)$ that are tangential at some point on the surface, then their normal curvature vectors at this point coincide. Denote by $\mathfrak{n}$(u,v) the unit normal vector of the surface, and call the inner product $_n(t)$ $\mathfrak{k}_n(t)$ $\mathfrak{n}(c(t))$ is called the normal curvature of the curve. Enter the commands of Animation 1 in the notebook Tazawa1.nb.

Key *Mathematica* functions : numerical calculation.
Key user commands : **mywireframe, normalcurvatureanim.**

2. Principal curvatures.

Let p be a point on a surface $f(u, v)$, and $\mathfrak{n}$ be the unit normal vector of the surfce at p . Consider the curve of the section of the surface by a plane including $\mathfrak{n}$, and orient it so that the unit principal normal vector of the curve coincides with $\mathfrak{n}$. The minimum k_1 and the maximum k_2 of the normal curvature of the section curve are called the principal curvatures. The mean curvature and the Gaussian curvature are defined by $H \quad \frac{k_1 \; k_2}{2}$ and $K \quad k_1 k_2$. Enter the commands of Animation 2 in the notebook Tazawa2.nb.

Key *Mathematica* functions : numerical culculation.
Key user commands : **planesectionanim22.**

3. Variation of a piece of a catenoid.

A surface with vanishing mean curvature is called a minimal surface. A minimal surface is characterized as the surface with the minimal area bounded by an assigned boundary. Animation 3 in Tazawa3.nb shows the change of the area under a variation of a piece of catenoid, a well known minimal surface.

Key *Mathematica* functions : **NIntegrate**.

4. Minimality of a geodesic.

The shortest curve jointing two given points on a surface is a geodesic. For an assigned point and direction, there is a unique geodesic, obtained by solving a system of ordinary differential equations. Animation 4 in Tazawa4.nb shows the change of the length under a variation of a geodesic on a randomly generated surface.

Key *Mathematica* functions : **NIntegrate, NDSolve**.
Key user commands : **mywireframe77.**

5. Approximation of a Geodesic

A geodesic is also characterized as a curve on a surface with vanishing geodesic curvature vector, which means, a geodesic looks infinitesimally like a line, if observed from the direction of the normal vector. Therefore, a geodesic can be approximated by jointing the pieces of curves that are the inverse images of segments on the tangent planes under the projections to the tangent planes. Animation 5 in Tazawa5.nb shows an approximation of the geodesic in Animation 4.

Key *Mathematica* functions : **NIntegrate, NSolve, NDSolve**.
Key user commands : **mywireframe77.**

6. Surfaces with Assigned fundamental quantities

The fundamental theorem of surfaces states that if a set of functions satisfy the integrability conditions of Gauss and Mainardi-Codazzi, then there exit surfaces whose first and second fundamental quantities coincide with these assigned functions, and they are unique up to isometries. Commands in Tazawa6.nb check the integrability conditions and plot the solution surface numerically.

Key *Mathematica* functions : **NIntegrate, NDSolve**.
Key user commands : **integrabilitycheck, numintegrabilitycheck, numplotfundsol**

References

Introduction to the theory of curves and surfaces with *Mathematica*, by Yoshihiko Tazawa, in preparation to be published in Japanese language from Addison-Wesley Publishers Japan