RECENT EXTENSIONS IN THE COMPUTATION OF OPTIMAL SIMPLIFIED MODELS FOR SYSTEMS WITH TIME DELAYS

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Abstract
The earlier optimal method [1,2] for the simplification of single variable systems having ordinary polynomial denominators has been extended to the multivariable case encompassing situations where the original or simplified model may have delays in their states. The performance index used for optimization is the integral of the time-weighted squared error between the responses of the original and simplified models. The performance index is first expressed in terms of the simplified model unknown parameters and a minimization of this index gives the simplified model optimal parameters. The results of this work find additional application in obtaining the closed-form expressions of the error squared integral between the reference input and output response for certain closed-loop systems involving plants with time delays.

Key words: Model simplification, Multivariable time delayed systems.

Introduction
The approximation of complex models by simple ones continues to generate the interest of systems engineers and scientists [3-6]. Approximation of high order systems by low order ones with time delays is common in process control [7-9]. Some classical graphical methods for achieving this are available[10]. One application of these simple models is in the implementation of internal model and Smith Predictor control schemes [11]. Another important purpose of the time delay is to ensure that both the complex and simplified models possess approximately equal amount of phase lags, at least, at critical frequencies.

On the other hand, it is sometimes necessary to accurately approximate a complex model with time delay by simpler model having no time delay. One common reason for this requirement is that the methods of process analysis and control system design are based on state variable techniques [12]. This paper considers the approximation of complex models with or without time delay by simpler ones with or without time delay. A general treatment is first given for multivariable systems having time delays both in the numerator and denominator terms. The results of this work facilitate the exact evaluation of the weighted error squared integral of systems involving quasi - rational transfer functions [7-9] as well as certain closed-loop systems where the original open loop systems possess time delays. The procedures are illustrated by examples.

It is important to compare this work with earlier ones. Halevi's[3] work deals with the reduction of high order systems without delay to low order ones with delay. His method of solution is elegant but he considered only impulse-like input. The sub-optimal method of Xue and Atherton[5] is simple and can be applied to single variable models having time delays in the denominator. It is however subject to error and inapplicable to the multivariable situation. Hwang and Chuang [13] considers only single variable problems where the high order model has no time delays. It is clear that this paper solves a larger class of problems than hitherto addressed. In general, capital letters denote the Laplace transform variable in this work..

Description of the Problem

Consider a stable linear time-invariant system with time delay whose input-output description is given by

$$Y(s) = G(s)\,U(s) = (G_{ij}(s))\,U(s) \qquad (1)$$

where U(s), Y (s) and G(s) are respectively the Laplace transforms of the input variable vector u(t), the output response vector y(t) and the n_y x n_u impulse response matrix g(t). The objective is to compute a stable simplified n_y x n_u delayed model $G_r(s)$ given by

$$Y_r(s) = G_r(s)\,U(s) = (G_{rij}(s))\,U(s) \qquad (2)$$

such that for a suitable input U(s) the integral of the weighted squared error between the output time responses of the original and reduced models

$$J = tr\left(\int_0^\infty \left[(h_{ij}(t)(y_{ij}(t) - y_{rij}(t)))\right]^T \left[(y_{ij}(t) - y_{rij}(t))\right]\right)dt \qquad (3)$$

is minimised, where y_{ij} and y_{rij} denotes ith output response to jth input change for the complex and simplified models respectively, tr (.) and superscript T denote respectively the trace and transpose of a matrix while $h_{ij}(t)$ is the weight for $e^2_{ij}(t)$ where

$$e_{ij}(t) = y_{ij}(t) - y_{rij}(t) \qquad (4)$$

The weight $h_{ij}(t)$ which is a suitable function of time bestows desirable properties on the response $y_{rij}(t)$. A weight which is a polynomial in t penalizes the error at long times and usually forces $y_{rij}(t)$ to approach $y_{ij}(t)$ speedily. The weight specified in this work takes the form

$$h_{ij}(t) = t^{k_{ij}} \qquad (5)$$

where k_{ij} is a positive integer. When every h_{ij} =1 in (3), then J is known as the integral of squared error, ISE. For the purposes of this work

$$U(s) = \frac{u_0}{s} + u_1 + \frac{u_2}{u_3 s + 1} \qquad (6)$$

where u_0, u_1, u_2 and u_3 are constants. The problem then is to compute $G_r(s)$ such that (3) is minimized subject to the constraint that all the poles p_i of $G_r(s)$ lie in the open left half plane,

$$Re(p_i) < 0 \qquad (7)$$

with zero steady state error in the output responses, that is

$$e(t) = (y_{ij}(t)-y_{rij}(t)) = 0 \text{ for } t \to \infty \qquad (8)$$

where $Re(p_i)$ denotes the real part of the complex number p_i. The steady state constraint (8) implies

$$G(0) = G_r(0) \qquad (9)$$

whenever $u_0 \neq 0$ in (6) thereby ensuring the existence of the indefinite integral (3). This is therefore a parameter optimization problem which can be efficiently solved by a suitable optimization procedure provided that the stability constraint (7) are satisfied during the course of the optimization and (3) can be cheaply computed. In order to cheaply compute (3), one makes use of Parseval's identity

$$J = tr\left(\int_0^\infty (t^{k_{ij}} e_{ij}^T(t))(e_{ij}(t))\right)dt = tr\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F(s)E(-s)\right)ds, i = \sqrt{-1}\,, \qquad (10)$$

where

$$E(s) = Y(s)-Y_r(s) = (G(s)-G_r(s))U(s) = (E_{ij}(s)) \quad (11)$$

and

$$F_{ij}(s) = (-1)^{k_{ij}} \frac{d^{k_{ij}} E_{ij}(s)}{ds^{k_{ij}}} \quad (12)$$

The frequency domain expression for J in (10) cannot be directly evaluated using recursive algorithms[14]. The introduction of a simple reformulation, as explained below, overcomes this problem.

Computation of the Integral of the Time-Weighted Squared Error

In order to make the treatment as general as possible, we consider the (i,j)th element of the error $E_{ij}(s)$ where the complex model has the form

$$G_{ij}(s) = \frac{H(s) + L(s)e^{-d_1 s}}{M(s) + L(s)e^{-d_2 s}} \quad (13)$$

and H(s),…,M(s) are polynomials in s of finite degree and of real coefficients while the (i,j)th element of the simplified model has the form

$$G_{rij}(s) = \frac{P(s)e^{-\tau s}}{Q(s)} \quad (14)$$

where P(s) and Q(s) are polynomials of finite degrees with real coefficients. Both $G_{ij}(s)$ and $G_{rij}(s)$ are assumed strictly proper. $E_{ij}(s)$ in (11) can be decomposed into its steady state and transient components

$$E_{ij}(s) = [(m_{ij}/s+Z(s))e^{-d_3 s}+Z_1(s)-(m_{ij}/s+Z_r(s))e^{-\tau s}] \quad (15)$$

where

$$m_{ij} = u_0 G_{ij}(0) \quad (16)$$

and

$$d_3 = d_1 - d_2 \quad (17)$$

We shall in the sequel, for simplicity, assume that $Z_1(s)$ is zero although the procedure set out below can easily be extended to situations where $Z_1(s)$ is non-zero.

In an earlier work[1] Taiwo has shown how, for the $Z_1(s) = 0$ case, J can be derived by expressing E_{ij} in the form (15). Drawing on this, it can be shown that

$$J = \sum_{i=1}^{n_y} \sum_{j=1}^{n_u} J_{ij} \quad (18)$$

$$J_{ij} = J_{aij} + J_{bij} + J_{cij} + J_{dij} + J_{eij} + J_{fij} \quad (19)$$

where

$$J_{aij} = \frac{m_{ij}^2}{k_{ij}+1}\left(\tau^{k_{ij}+1} - (d_1 - d_2)^{k_{ij}+1}\right) \quad (20)$$

$$J_{bij} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} X(s)Z(-s)e^{(d_1-d_2)s} ds \quad (21)$$

where $X(s) = (-1)^{kij} \frac{d^{k_{ij}}}{ds^{kij}}\left(Z_r(s)e^{-\tau s}\right)$ (22)

An explanation of how to evaluate (21) employing residue theorem is given in the Appendix.

$$J_{cij} = \frac{1}{2\pi j}\int_{-i\infty}^{i\infty} X_r(s)Z_r(-s)e^{\tau s} ds \quad (23)$$

Where $X_r(s) = (-1)^{kij} \frac{d^{k_{ij}}}{ds^{kij}}\left(Z_r(s)e^{-\tau s}\right)$ (24)

$$J_{dij} = -\frac{1}{\pi i}\int_{-i\infty}^{i\infty} P(s)Z_r(-s)e^{-\tau s}ds \quad (25)$$

where $P(s) = (-1)^{kij} \frac{d^{k_{ij}}}{ds^{kij}}\left(W(s)e^{-\tau s}\right)$ (26)

Note that the transformation from z to w is to impose the larger time delay τ on this function thereby facilitating (25) and its evaluation.

$$J_{eij} = 2m_{ij} X(0) \quad (27)$$

and

$$J_{fij} = -2m_{ij}P(0) \quad (28)$$

The changes needed in the above formula are apparent for the situation when $d_1 - d_2 > \tau$ and detailed explanations form similar situations are given elsewhere[1,2]. From the foregoing, it is clear that the minimization of J facilitates the computation of the optimal $G_r(s)$.

Illustrative Examples

Example 1. For approximating complex distributed systems or systems with a large pole-zero excess, Lepschy et al [9] propose the four parameter model of the form

$$G_r(s) = k_1e^{-\tau s} \Big/ (1 + sd_1 + k_2\, e^{-\tau s}) \quad (29)$$

as a simple model for the process

$$G(s) = 1 \Big/ (1 + s + 0.9s^2 + 0.44s^3 + 0.14s^4 + 0.035s^5) \quad (30)$$

The parameters of (29) obtained using moment matching together with the J for h(t) =1 are given in Table 1. In this work we consider the simpler three-parameter model

$$G_r(s) = ke^{-\tau_1 s} \Big/ (s + ke^{-\tau_2 s}) \quad (31)$$

to approximate (30). Following the development in this paper, and with $u_0=1$, $u_1 = u_2 = u_3 = 0$, we have for this example, m = 1 = G(0). $J_a = \tau_1 - \tau_2$, $J_b = 1.729529$, $J_c = \cos k\tau_2 / (2k(1 - \sin k\tau_2))$, $J_e=2Z(0)$,

$J_d = -2\sum_{i=1}^{5} a_i e^{-b_i(\tau_1-\tau_2)}/(b_i + ke^{-\tau_2 b_i})$, $J_f = -2\sum_{i=1}^{5} a_i\, e^{-(\tau_1-\tau_2)b_i} \big/ b_i$, where the a_i and $-b_i$ are the residues and poles of G(s). Upon choosing h(t) = 1 in (3) the technique of this paper is used to compute a simplified model with less than ¼ of the J obtained with moment matching employing model (29). We also employed moment matching to compute suitable parameters for model (31). As indicated in Table 1, the J is still more than four times the value obtained using optimization. In order to compare the effectiveness of the above $G_r(s)$ vis a vis the second order plus time delay model, this latter model in the form

$$G_r(s) = (c_1s+1)\, e^{-\tau_3 s} \big/ (d_2s^2 + d_1s + 1) \quad (32)$$

was tested using both optimization and moment matching. While the moment matching results indicate the superiority of the model (29) having time delay in the denominator, optimization indicates that the model (32) without delay in the denominator is superior for this example.

Table 1. J for the various approximants of example 1.

Method	Simplified Model Parameters				J
	k_1 or k or τ_3	k_2 or c_1	τ_1 or d_1	τ_2 or d_2 or τ	
This work using model (31)	1.24488	-	1.128126	0.83682	0.0199004
Moment matching using model (31)	1.09535	-	0.981666	0.894615	0.0490344
Moment matching [9] using model (29)	4.1924	3.1924	3.1669	1.0254	0.0480128
Moment matching using model (32)	0.536555	-0.1484086	0.3150362	0.4386114	0.055243
This work using model (32)	0.859226	-	0.237378	0.3574728	0.01816

Example 2 Evaluation of the ISE for a first order plus time delay system when feedback controller has been parametrized using internal model control (IMC) is important in various applications. Consider the plant $G(s) = c\,e^{-\tau s}/(s+c)$. With a first order filter having a time constant λ the classical feedback controller parametrized using IMC is given by $(\frac{s}{c}+1)\Big/(\lambda s+1-e^{-\tau s})$. Assuming the reference input is a unit step change, then the error is given by E(s) = $(\lambda s+1-e^{-\tau s})/(s(\lambda s+1))$. Using the theory of this paper, it is easily found that the ISE is given by $J = \tau+\lambda/2$.

Example 3. A model of an air compressor has two inputs (guide-vane and blow-off valve signals) and two outputs (pressure and flow). The transfer function[6] is

$$G(s) = \begin{bmatrix} \dfrac{0.1133e^{-0.715s}}{1.783s^2+4.48s+1} & \dfrac{0.9222}{2.071s+1} \\ \dfrac{0.3378e^{-0.299s}}{0.361s^2+1.09s+1} & \dfrac{-0.321e^{-0.94s}}{0.104s^2+2.463s+1} \end{bmatrix} \qquad (33)$$

G(s) is non-rational, and it is assumed that an accurate rational approximant is desired. This can be the case when a design technique is only applicable to rational systems, expressible in the usual state variable form. Note that the minimum number of differential equations needed to simulate this plant is seven. If it is desired to reduce this model to second order using optimization, one may assume the reduced model expressed in the controllable form:

$$\dot{x}(t) = A_x(t) + B_r u(t), \qquad y = C_r x(t) \qquad (34)$$

with

$$A_r = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B_r = I_2,\ C_r = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, G_r(s) = C_r(sI - A_r)^{-1} B_r. \qquad (35)$$

The computation of $G_r(s)$ can be done with pencil and paper or by using any software with symbolic computation capabilities. Assuming $u_o = 1$, $u_1 = u_2 = u_3 = 0$, the technique in the previous section can be used to obtain J_{ij} for each error term $E_{ij}(s)$ in terms of the unknown parameters in equation (35). Hence one can obtain J in terms of these parameters. Setting h(t) = 1, the optimal parameters of obtained are $A_r = \begin{bmatrix} -0.5644 & -0.2994 \\ 0.09037 & -0.4126 \end{bmatrix}, C_r = \begin{bmatrix} -0.01939 & 0.4145 \\ 0.21965 & -0.03123 \end{bmatrix}$

Table 2 shows that this model is favourable in terms of J to other second order reduced models hitherto obtained.

Further work was done to obtain a third order reduced model with

$$A_r = \begin{bmatrix} 0 & 0 & -a_{13} \\ 1 & 0 & -a_{23} \\ 0 & 1 & -a_{33} \end{bmatrix}, B_r = \begin{bmatrix} 1 & b_{12} \\ 0 & b_{22} \\ 0 & b_{32} \end{bmatrix}, C_r = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

With similar weight and input used above, the theory of this paper may be used to obtain the optimal parameters of the reduced model as given in Table 2. Observe that unlike moment matching which sometimes gives unstable reduced model, our method always gives stable reduced models. Our method may also be used to compute a reduced model of any order unlike the classical moment matching method when applied to multivariable process. When optimal values of relatively many parameters are desired, it may not be easy to know a good starting point. One method of overcoming this problem is to compute a high order moment approximant, obtain a balanced realization of desired order and transform the latter to the canonical form given above. This usually gives a good starting point which will facilitate speedy computation of the reduced model optimal parameters. One important advantage optimal reduced models is that they can be made to satisfy several desirable properties and constraints which may be in-built during the computation.

Table 2. J for various approximants of Example 3.

Method	J
This work, second order model	0.008711
Moment matching[6] about s=0 and s= 0.1165	0.0095117
Moment matching[6] about s=0 only	0.01034
This work, third order model	0.00246617

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Appendix

Evaluation of integral (21) [15]

Z (s) will in general take the form

$$Z(s) = \frac{B(s) + D(s)e^{-s\tau_1}}{A(s) + C(s)e^{-\tau s}} \quad \text{(A1)}$$

where A(s), …, D(s) are polynomials of finite degree with real coefficients. With h(t)=1

$$J_{bij} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \left(\frac{B(s) + D(s)e^{-s\tau_1}}{A(s) + C(s)e^{-s\tau}}\right)\left(\frac{B(-s) = D(-s)e^{s\tau}}{A(-s) + C(-s)e^{s\tau}}\right) ds \quad \text{(A2)}$$

For delay free systems it is possible to evaluate the integral by closing the contour on either the left or the right and using the theory of residues. Such an approach cannot work here because there are in general an infinite number of poles in both the left and right half planes. However, by first suitably rearranging the integral in such a way that there are only a finite number of relevant poles it is possible to evaluate (A2). The basic idea is to split the integral into two parts, the first of which consists of all the poles arising from the zeros of $A(s) + C(s)e^{-\tau s}$ and the second all those arising from the zeros of $A(-s) + C(-s)\,e^{\tau s}$. This is achieved by first obtaining an equivalent form for Z(-s) at the poles of Z(s) as follows. At such points

$$A(s) + C(s)e^{-\tau s} = 0 \quad \text{(A3)}$$

and so

$$Z(-s) = \frac{B(-s) + D(-s)e^{s\tau_1}}{A(-s) + C(-s)e^{\tau s}} = \frac{B(-s)A(s) - D(-s)C_1(s)}{A(-s)A(s) - C(-s)C(s)} \quad \text{(A4)}$$

where $C_1(s) = C(s)e^{-s(\tau_1 - \tau)}$

The desired rearrangement is then given by

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\frac{B(s)+D(s)e^{-s\tau_1}}{A(s)+C(s)e^{-s\tau}}\right)x\left(\frac{B(-s)+D(-s)e^{s\tau_1}}{A(-s)+C(-s)e^{s\tau}} - \left(\frac{B(-s)A(s)-D(-s)C_1(s)}{A(-s)A(s)-C(-s)C(s)}\right)\right)$$
$$+\frac{B(s)+D(s)e^{-s\tau_1}}{A(s)+C(s)e^{-s\tau}}x\left(\frac{B(-s)A(s)-D(-s)C_1(s)}{A(-s)A(s)-C(-s)C(s)}\right)ds \quad \text{(A5)}$$

which may be written in the alternative form

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\frac{B(s)+D(s)e^{-s\tau_1}}{A(-s)A(s)-C(-s)C(s)}\right)x\left(\frac{(D(-s)A(-s)-B(-s)C(-s))e^{s\tau}}{A(-s)+C(-s)e^{s\tau}} + \left(\frac{B(-s)A(s)-D(-s)C_1(s)}{A(-s)A(s)-C(-s)C(s)}\right)\right)d \quad \text{(A6)}$$

These suggest that the integral may be evaluated by closing the contour in the left half- plane for the first part of the integral since all the zeros of [A(-s) + C(-s)$e^{s\tau}$] lie in the right half-plane and by closing the contour in the right half-plane for the second part of the integral since all the zeros of [A(s) + C(s)$e^{-s\tau}$] lie in the left half-plane. The poles arising from the roots of

$$A(-s)A(s) - C(-s)C(s) = 0 \quad \text{(A7)}$$

must also be considered. Since some of these zeros may lie on the imaginary axis, it may be necessary to first indent the contour (to the left half plane) around these points before separating the integral into two parts and closing the respective contours.

It is now possible to close the contour in the left half-plane for the first part of (A6), and in the right half-plane for the second part of (A6). In both cases, the only enclosed poles arise from the enclosed zeros of (A7). Assuming that the integrals round the semicircles at infinity are zero (as will be so in most cases of practical interest), it follows from (A6) that

$$J = -\sum_k \operatorname*{res}_{s=s_k}\frac{B(s)+D(s)e^{-s\tau_1}}{A(s)+C(s)e^{-\tau s}}\left(\frac{B(-s)A(s)-D(-s)C_1(s)}{A(-s)A(s)-C(-s)C(s)}\right) \quad \text{(A8)}$$

where the s_k are the roots of (A7).

For the general case in which e(t) is weighted, the above procedure may be repeated, with no additional complications, and it is found that

$$J_1 = -\sum_k \operatorname*{res}_{s=s_k}\left(\frac{\bar{B}+\bar{D}e^{s\tau_1}}{\bar{A}+\bar{C}\exp(s\tau)}\right)x\left\{\frac{B'\bar{A}-D'\bar{C}+\tau_1 D\bar{C}}{A\bar{A}-C\bar{C}} - \frac{\left(B\bar{A}-D\bar{C}\right)\left(A'\bar{A}-C\bar{C}+\tau C\bar{C}\right)}{\left(\bar{A}A-C\bar{C}\right)^2}\right\} \quad \text{(A9)}$$

in which, exactly as in (A8), the summation is taken over the solutions s = s_k of the polynomial equation (A7). In (6) the prime denotes differentiation, and the shorthand notation $\bar{A}$ = A(-s), etc. is used.