NORMAL LINES DRAWN TO ELLIPSES AND ELLIPTIC INTEGRALS

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Abstract

Consider the ellipse $x^2/a^2 + y^2/b^2 = 1$ where $a > b > 0$, and an arbitrary point $P_0(x_0, y_0)$ on the plane. Finding the minimum distance from P_0 to the ellipse is a well-known problem. One can easily show that the minimum distance path lies along a normal line to the ellipse, passing through P_0. This paper deals with a study of all the normal lines drawn from point P_0 to the ellipse. We have used the computer algebra system *Mathematica* to illustrate and discover several aspects of these normal lines. *Mathematica* can be used in contemporary mathematical research and educaiton in more than one way: as a computational, visualization, experimentation and a conjecture forming tool (see [4]-[14]). The paper well illustrates such usage via a study of the normal lines to the ellipse. We used *Matheamtica* version 3.0 on a *Windows* 95 platform. Some good references on *Mathematica* are [2], [16], [18] and [19].

1. Introduction

Consider the ellipse given by the following equation, where $a > b > 0$:

$$x^2/a^2 + y^2/b^2 = 1 \qquad (1.1)$$

Suppose $P_0(x_0, y_0)$ is an arbitrary point on the plane. In this introductory section of the paper, we will investigate on the number of normal lines that can be drawn from P_0 to the ellipse. Most of the material in the introduction can also be found in [14], but in order to make this paper self contained as much as possible, we will also include it here.

Suppose P_0P is a normal line drawn from P_0 to the ellipse (1.1), meeting the ellipse at the point $P(a\cos\theta, b\sin\theta)$ where $0 \le \theta < 2\pi$.

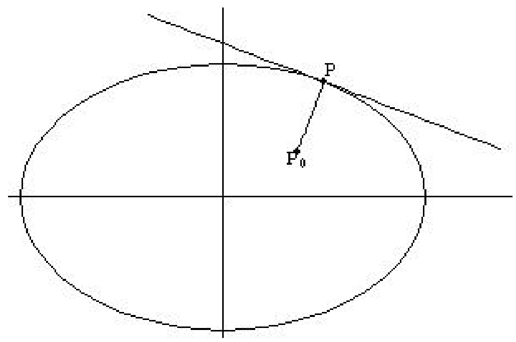

Fig 1.1. Normal Line to an Ellipse

One can find the slope of the line P_0P in two different ways:

$$\text{slope of } P_0P = \frac{y_0 - b\sin\theta}{x_0 - a\cos\theta} \qquad (1.2)$$

One can implicitly differentiate the equation (1.1) with respect to x, to obtain the slope of the tangent line at (x, y):

$$dy/dx = -b^2x/(a^2y) \qquad (1.3)$$

One can evaluate equation (1.3) at $P(a\cos\theta, b\sin\theta)$ to find the slope of the tangent line to the ellipse at P. The slope of the normal line P_0P at P can be obtained by taking the negative reciprocal of this:

$$\text{slope of } P_0P = \left.\frac{a^2y}{b^2x}\right|_{(a\cos\theta,\, b\sin\theta)} = \frac{a\sin\theta}{b\cos\theta} \qquad (1.4)$$

One can set the right-hand sides of equations (1.2) and (1.4) equal to each other and simplify to obtain the following equation:

$$x_0(a\sin\theta) - y_0(b\cos\theta) = (a^2 - b^2)\sin\theta\cos\theta \qquad (1.5)$$

It is important to realize that for given values a, b, x_0 and $y_0,$ the number of distinct solutions for t of the above equation (1.5), where $0 \le t < 2\pi$, correspond to the number of distinct normal lines that can be drawn from $P_0(x_0, y_0)$ to the ellipse. In order to solve the equation (1.5), we will consider two cases:

<u>Case 1</u>: $y_0 = 0$

In this case, the equation (1.5) will imply that $\sin t = 0$ or $\cos t = ax_0/(a^2 - b^2).$ The equation $\sin t = 0$ implies that $P = (a, 0)$ or $(-a, 0).$ Therefore, these two values for P will correspond to two distinct normal lines (along the x-axis). However, the equation $\cos t = ax_0/(a^2 - b^2)$ will produce real values for t if and only if $|x_0| \le (a^2 - b^2)/a = ae^2$ where $e = \sqrt{a^2 - b^2}/a$ is the eccentricity of the ellipse. If $|x_0| < (a^2 - b^2)/a,$ then we obtain two distinct values of t such that $\cos t = ax_0/(a^2 - b^2)$ with $0 \le t < 2\pi$ and $t \ne \pi$, corresponding to two more distinct normal lines. On the other hand, if $|x_0| = (a^2 - b^2)/a$, then we obtain the previous normal lines P_0P along the x-axis with $P = (a, 0)$ or $(-a, 0).$ Hence one can conclude that the case $y_0 = 0$ corresponds to two or four distinct normal lines according as $|x_0| \ge ae^2$ or $|x_0| < ae^2.$

<u>Case 2</u>: $y_0 \ne 0$

This case implies, via equation (1.5) that $t \ne \pi.$ Therefore, $\tan(t/2)$ is well defined. Therefore, let us use the trigonometric substitution $z = \tan(t/2)$ to solve equation (1.5) (see [3]). One can easily show that $\sin t = 2z/(1 + z^2)$ and $\cos t = (1 - z^2)/(1 + z^2).$ Substitute these expressions back in equation (1.5) and simplify to obtain

$$z^4(by_0) + 2z^3[ax_0 + (a^2 - b^2)] + 2z[ax_0 - (a^2 - b^2)] - by_0 = 0 \qquad (1.6)$$

The "**Solve**" command of *Mathematica* can certainly solve the above equation (1.6) for **>**. However, as the reader can verify, these solutions are almost useless, because of their complexity. Rather than the actual solutions themselves, at this point we are more interested in the nature or the number of solutions. Therefore, we will proceed as follows:

One can further simplify equation (1.6). Since +# • ,# œ +/#ß the equation (1.6) will read

>%Ð,C!Ñ € #+>$ÐB! € +/#Ñ € #+>ÐB! • +/#Ñ • ,C! œ !   Ð"Þ(Ñ

Since C! Á !, the above equation (1.7) represents a quartic equation with real coefficients. Notice that we have transformed the trigonometric equation (1.5) to a polynomial equation (1.7). The discriminant of the quartic equation (1.7) reveals the nature of its roots.

Recall that the discriminant of the quartic 9ÐBÑ œ +!B% € %+"B$ € '+#B# € %+$B€+% with roots :ß ;ß < and = are given by (see [1])

. œ +!'Ð: • ;Ñ#Ð: • <Ñ#Ð: • =Ñ#Ð; • <Ñ#Ð; • =Ñ#Ð< • =Ñ#   Ð"Þ)Ñ

One can also obtain the following version for the discriminant in terms of the coefficients of the quartic (see [1]):

. œ #&'ÒÐ+!+% • %+"+$ € $+##Ñ$ • #(Ð+!+#+% € #+"+#+$ • +!+$# • +%+"# • +#$Ñ#Ó   Ð"Þ*Ñ

Using the Intermediate Value Theorem in calculus, one can easily see that our equation (1.7) has at least two distinct real roots (see [17]). Call these two real roots ! and ", and the other two roots # and $. The equation (1.8) implies that the discriminant **H** of the quartic (1.7) is given by

H œ Ð,C!Ñ'Ð! • "Ñ#Ð! • #Ñ#Ð! • $Ñ#Ð" • #Ñ#Ð" • $Ñ#Ð# • $Ñ#   Ð"Þ"!Ñ

Case (a): Suppose the equation (1.7) has exactly two distinct real roots. Then

Ð3Ñ # and $ are real Ê H œ ! Ð33Ñ # and $ are non real Ê H • !

Case (b): Suppose the equation (1.7) has exactly three distinct real roots. Then clearly

H œ !þ

Case (c): Suppose that the equation (1.7) has exactly four distinct real roots. Then

clearly H ž !þ

Let us now calculate H in terms of the coefficients of (1.7). By comparing the polynomial 9ÐBÑ œ +!B% € %+"B$ € '+#B# € %+$B€+% with the left-hand side of equation of (1.7), one finds that +! œ ,C!ß +" œ +ÐB! € +/#ÑÎ#ß +# œ !ß +$ œ +ÐB! • +/#ÑÎ#ß and +% œ • ,C!þ Therefore, equation (1.9) implies that

H œ HÐB!ß C!Ñ œ #&'ÖÒÐ+# • ,#Ñ# • +#B!# • ,#C!#Ó$ • #(+#,#Ð+# • ,#ÑB!#C!#× Ð"þ""Ñ

One can use *Mathematica* for simplification purposes to arrive at the above equation. Depending on the values of B! and C!ß HÐB!ßC!Ñ could be positive (i.e. 4 distinct normals), negative (i.e. 2 distinct normals) , or zero (i. e. 2 or 3 normals). To find out when does this happen, one can draw the graph of the following equation:

0ÐBßCÑ œ ÒÐ+# • ,#Ñ# • +#B# • ,#C#Ó$ • #(+#,#Ð+# • ,#ÑB#C# œ ! Ð"þ"#Ñ

For given specific values of + and ,ß one can use the "**ImplicitPlot**" command of *Mathematica* to see the shape of the graph given by equation (1.12). In general, one can show that the equation (1.12) has the following parametrization:

B œ ÒÐ+# • ,#ÑG9=$! ÓÎ+ and C œ ÒÐ+# • ,#ÑW38$! ÓÎ, ; ! Ÿ ! • #1 Ð"þ"$Ñ

For + œ ' and , œ %, the following *Mathematica* command "**ParametricPlot**" produces the graph of equation (1.13):

a=6; b=4;

ParametricPlot[{(a^2-b^2)Cos[alpha]^3/a,(a^2-b^2)Sin[alpha]^3/b},

{alpha,0,2Pi}, AspectRatio->Automatic, Ticks->None]

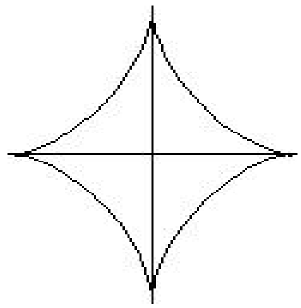

Fig. 1.2. The Graph of Equation (1.13): Generalized Astroid

Observe that the above graph is quite similar to the astroid $x^{2/3} + y^{2/3} = r^{2/3}$ where r is a constant (see [20]). Motivated by this, it is not hard to show that any of the equations (1.12) or (1.13) is equivalent to the following equation, which we will refer to as the "generalized astroid":

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3} \qquad (1.14)$$

Using the above graph, one sees that if (x_0, y_0) is inside the curve, then $H(x_0, y_0)$ is positive; If (x_0, y_0) is on the curve, then $H(x_0, y_0)$ is equal to zero; If (x_0, y_0) is outside the curve, then $H(x_0, y_0)$ is negative.

One can now combine the outcomes of cases 1 and 2 in this section to arrive at the following conclusion: If P_0 is inside the generalized astroid, then one can draw four distinct normal lines from P_0 to the ellipse; If P_0 is outside the generalized astroid, then one can draw two distinct normal lines from P_0 to the ellipse;

If $T_!$ is on the generalized astroid but not a cusp point, then one can draw three distinct normal lines from $T_!$ to the ellipse; If $T_!$ is one of the four cusp points on the generalized astroid, then one can draw two distinct normal lines from $T_!$ to the ellipse. This is illustrated by the following figure.

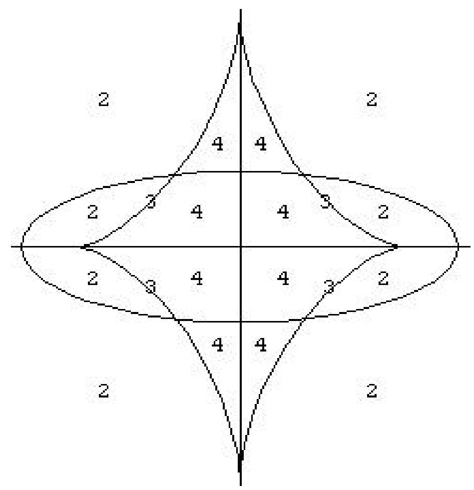

Fig. 1.3. The Number of Normal Lines From Points on the Plane to the Ellipse

2. *Mathematica* Generation of the Normal Lines

One can use *Mathematica* to graph the normal lines at different points on the ellipse. The following programs draws these normals, along with the graphs of the ellipse (1.1) and the modified astroid (1.13):

Program 2.1

```
<<Graphics`ImplicitPlot`
a=6;b=4;
p1=ImplicitPlot[x^2/a^2+y^2/b^2==1,{x,-a,a},PlotStyle->{RGBColor[1,0,0]},
        AspectRatio->Automatic,DisplayFunction->Identity]
```

```
p2=ParametricPlot[{((a^2-b^2)/(a))(Cos[theta])^3,(((a^2-b^2))/(b))(Sin[theta])^3},
        {theta,0,2Pi},
        PlotStyle->{Thickness[1/80],RGBColor[0,0,1]},AspectRatio->Automatic,
        DisplayFunction->Identity]
p3=ImplicitPlot[ Evaluate[Table[ y-b*Sin[t]==a*Sin[t] (x-a*Cos[t])/(b*Cos[t]),
        {t,0,2Pi,2Pi/100}]],{x,-a,a},AspectRatio->Automatic,
        DisplayFunction->Identity]
Show[{p1,p2,p3},AspectRatio->Automatic,PlotRange->{{-a,a},{-b-2,b+2}},
       DisplayFunction->$DisplayFunction]
```

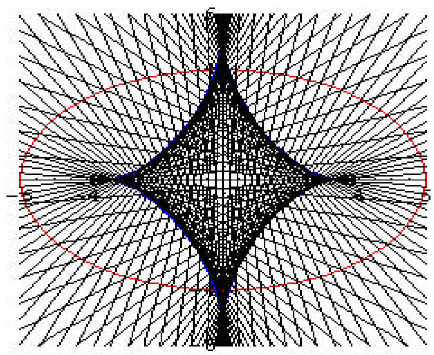

Fig. 2.1. The Generalized Astroid as the Envelope of the Normals to the Ellipse

The above figure suggests that each normal line would touch the generalized astroid (1.13). It is well-known that it is the case. In other words, one can easily show that the envelope of all the normal lines to the ellipse is precisely the generalized astroid given by (1.13). By definition, the generalized astroid is called the evolute of the ellipse (see [3] and [20]).

The above program (2.1) only creates a static picture of the normal lines at points on the ellipse. However, one can use *Mathematica* to create an animation of the normal lines at points **T** on the ellipse, as **T** traverses the ellipse. Such animations in general serve as better visualization tools in contemporary mathematics instruction. The following *Mathematica* program creates our first animation of the normal lines:

Program 2.2

```
<<Graphics`ImplicitPlot`
a=6;b=4;
p1=ImplicitPlot[x^2/a^2+y^2/b^2==1,{x,-a,a},PlotStyle->{RGBColor[1,0,0]},
      AspectRatio->Automatic,DisplayFunction->Identity]
p2=ParametricPlot[{((a^2-b^2)/(a))(Cos[theta])^3,(((a^2-b^2))/(b))(Sin[theta])^3},
     {theta,0,2Pi},
     PlotStyle->{Thickness[1/100],RGBColor[0,0,1]},AspectRatio->Automatic,
     DisplayFunction->Identity]
Do[Show[{p1,p2,ImplicitPlot[ Evaluate[Table[
       y-b*Sin[t]==a*Sin[t] (x-a*Cos[t])/ (b*Cos[t]), {t,0,s,2Pi/41}]],
      {x,-a,a},DisplayFunction->Identity]},
       PlotRange->{{-a,a},{-((a^2-b^2)/b)^(3/2),((a^2-b^2)/b)^(3/2)}},
       DisplayFunction->$DisplayFunction],
                                        {s,0,2Pi,2Pi/41}]
```

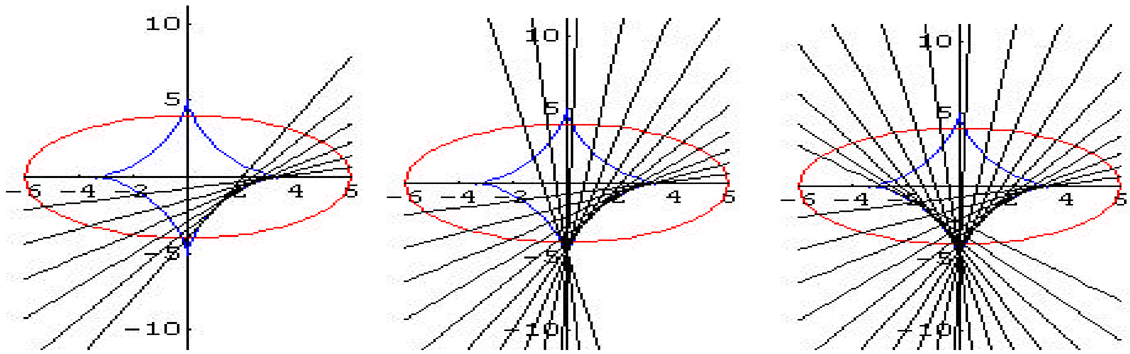

Fig. 2.2. An Animation of Normals at Various Points on the Ellipse

In the previous section, we noticed that the depending on the position of the point T¡ÐB¡ßC¡Ñ on the plane, one can draw either two, three or four distinct normal lines to the ellipse. The following *Mathematica* program based on numerical solution of the quartic equation (1.6) will draw all the normal lines to the ellipse (1.1) from an arbitrary point T¡ÐB¡ßC¡Ñ on the plane:

Program 2.3

```
StylePrint["THE NORMAL LINES FROM A POINT ON THE PLANE TO AN
ELLIPSE", "Text",FontSize->24,FontFamily->"Times", FontColor->RGBColor[1,0,0],
FontWeight->"Bold",TextAlignment->Center ]
<<Graphics`ImplicitPlot`
a=?; b=?; x0=? ; y0=? ;
expr=((a^2-b^2)^2-a^2*x0^2-b^2*y0^2)^3-27a^2*b^2*(a^2-b^2)^2*x0^2*y0^2;
x1=a^2*x0((a^2-b^2)^2*y0^2-b^6)/(x0^2*b^6+y0^2*a^6);
y1=b^2*y0((a^2-b^2)^2*x0^2-a^6)/(x0^2*b^6+y0^2*a^6);
roots=t/.NSolve[t^4*b*y0+t^3(2a*x0+2(a^2-b^2))+t(2a*x0-2(a^2-b^2))-b*y0==0, t];
list1=Cases[roots,x_/; Im[x]==0];
list2=Table[{t->list1[[i]]},{i,1,Length[list1]}];
```

```
list3={a*(1-t^2)/(1+t^2),b*2t/(1+t^2)}//.list2;
Which[y0==0,list=Union[list3,{{-a,0}}],
        y0!=0 && x0^2/a^2+y0^2/b^2!=1,list=list3,y0!=0 && x0^2/a^2+y0^2/b^2==1,
        list=Union[list3,N[{{x1,y1}}]]];
complist=Complement[list,N[{{x1,y1}}]];
l=Length[list];
p=ParametricPlot[{((a^2-b^2)/(a))(Cos[theta])^3,(((a^2-b^2))/(b))( Sin[theta])^3},
        {theta,0,2Pi}, PlotStyle->{Thickness[1/180],RGBColor[0.2,0.1,0.5]},
        DisplayFunction->Identity]
Show[{ImplicitPlot[x^2/a^2+y^2/b^2==1,{x,-a,a},PlotStyle->{RGBColor[1,0,0]},
    AspectRatio->Automatic,
   Epilog->{{PointSize[1/60],RGBColor[1,0,0],Point[{x0,y0}]},
  Table[{PointSize[1/90],RGBColor[0,0,1], Point[complist[[i]]] },{i,1,Length[complist]}],
  Table[{RGBColor[0.1,1,0],Thickness[1/300],Line[{{x0,y0},list[[i]]}]},{i,1,l}]},
   DisplayFunction->Identity],p},DisplayFunction->$DisplayFunction]
Which[(expr== 0  &&  ({x0,y0}=={(a^2-b^2)/a,0}||{x0,y0}=={-(a^2-b^2)/a,0}
        ||{x0,y0}=={0,(a^2-b^2)/b}||{x0,y0}=={0,-(a^2-b^2)/b}))
        ||expr<0, StylePrint["TWO  NORMAL  LINES", "Text",FontSize->36,
    FontFamily->"Bodoni MT Ultra Bold", FontWeight->"Bold",TextAlignment->Center,
    Background->RGBColor[0.8,0,0.75] ],
    expr==0,StylePrint["THREE  NORMAL  LINES", "Text",FontSize->36,
    FontFamily->"Bodoni MT Ultra Bold", FontWeight->"Bold",TextAlignment->Center,
    Background->RGBColor[0.8,0,0.75] ],
    expr>0,StylePrint["FOUR  NORMAL  LINES", "Text",FontSize->36,
    FontFamily->"Bodoni MT Ultra Bold", FontWeight->"Bold",TextAlignment->Center,
    Background->RGBColor[0.8,0,0.75] ]]
```

In order to execute the program enter values for **+ß ,ß B! and C!** in the above program, click anywhere on the program lines, and press "**Shift-Enter**". The following is a sample of such outputs:

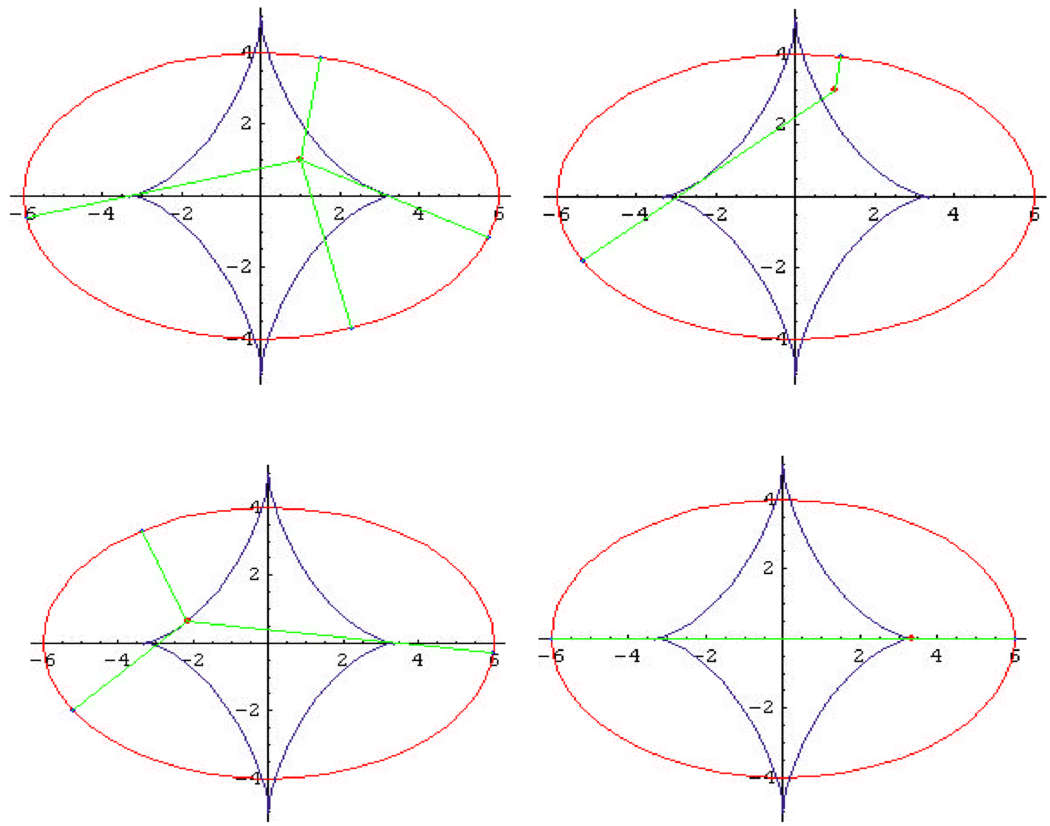

Fig. 2.3. Normal Lines from Various Points on the Plane to the Ellipse

We also used *Mathematica* to create another animation of these normal lines. These animations serve as excellent visualization devices to better understand the normal lines drawn to an ellipse. The following program creates an animation of the normal lines as a point travels along an "amplified sine curve" **0ÐBÑ œ #BW38Ò#ÐB • #ÑÓ**. As the point travels along this curves it repeatedly enters and exits the region bounded by the generalized astroid. In the process, one observes a varying number of normal lines from the moving point to the ellipse.

Program 2.4

```
<<Graphics`ImplicitPlot`
a=6; b=4;
f[x_]:=2x*Sin[2(x-2)]
p1=ParametricPlot[{((a^2-b^2)/(a))(Cos[theta])^3,(((a^2-b^2))/(b))( Sin[theta])^3},
      {theta,0,2Pi}, PlotStyle->{Thickness[1/150],RGBColor[0.2,0.1,0.5]},
       DisplayFunction->Identity]
p2=Plot[f[x],{x,-4,4},PlotStyle->{Thickness[1/180],RGBColor[0,0,1]},
             DisplayFunction->Identity];
Do[{x0=s;y0=f[s];
      x1=a^2*x0((a^2-b^2)^2*y0^2-b^6)/(x0^2*b^6+y0^2*a^6);
      y1=b^2*y0((a^2-b^2)^2*x0^2-a^6)/(x0^2*b^6+y0^2*a^6);
      roots=t/.NSolve[t^4*b*y0+t^3(2a*x0+2(a^2-b^2))+t(2a*x0-2(a^2-b^2))-b*y0==0, t];
      list1=Cases[roots,x_/; Im[x]==0];
      list2=Table[{t->list1[[i]]},{i,1,Length[list1]}];
      list3={a*(1-t^2)/(1+t^2),b*2t/(1+t^2)}//.list2;
      Which[y0==0,list=Union[list3,{{-a,0}}],
           y0!=0 && x0^2/a^2+y0^2/b^2!=1,list=list3,y0!=0 && x0^2/a^2+y0^2/b^2==1,
           list=Union[list3,N[{{x1,y1}}]]]; complist=Complement[list,N[{{x1,y1}}]];
          l=Length[list];
       Show[{ImplicitPlot[{x^2/a^2+y^2/b^2==1},{x,-a,a},
       PlotStyle->{{Thickness[1/110],RGBColor[1,0,0]}}, AspectRatio->Automatic,
       Epilog->{{PointSize[1/60],RGBColor[1,0,0],Point[{x0,y0}]},
  Table[{PointSize[1/90],RGBColor[0,0,1], Point[complist[[i]]] },{i,1,Length[complist]}],
  Table[{Thickness[1/200],Line[{{x0,y0},list[[i]]}]},{i,1,l}]},
```

DisplayFunction->Identity],p1,p2},DisplayFunction->$DisplayFunction,

PlotRange->{{-a,a},{-7.5,7.5}}] }, {s,-4,4,8/40}]

A few frames of the animation are given below:

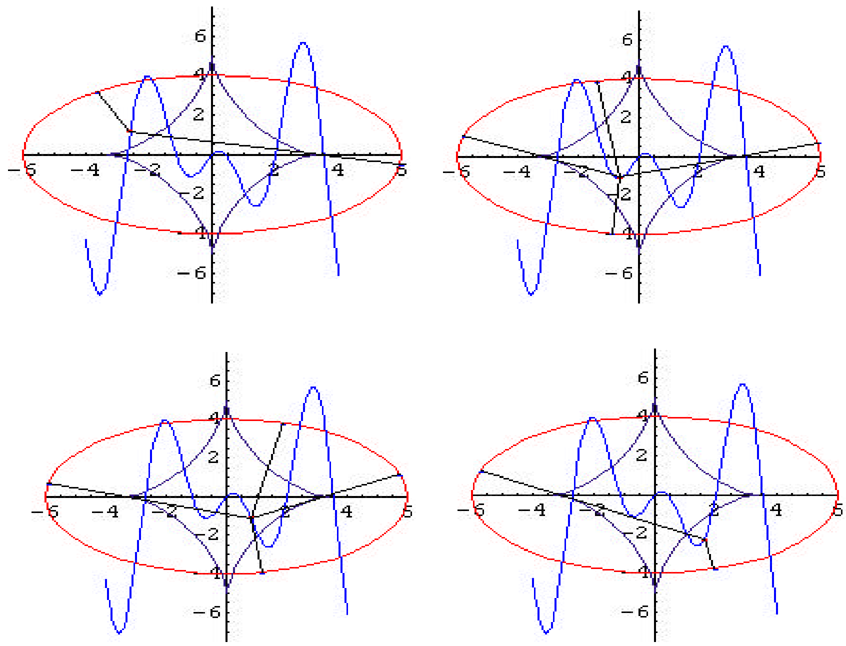

Fig. 2.4. An Animation of Normal Lines

3. Intersections of the Ellipse and its Evolute

One can now ask whether the ellipse given by equation (1.1) will intersect its evolute (i.e. the generalized astroid) given by either of the equations (1.13) or (1.14). One can use the *Mathematica* substitution command "**/.**" to simultaneously solve the equations (1.1) and (1.13):

x^2/a^2+y^2/b^2= =1 /. {x->(a^2-b^2)Cos[alpha]^3/a,y->(a^2-b^2)Sin[alpha]^3/b}

Hence one can obtain the following equation, where **? œ W38# ! À**

Ð" • ?Ñ$ / +% € ?$ / ,% œ " / Ð+# • ,#Ñ#   Ð$Þ"Ñ

One can use the *Mathematica* "**Solve**" command to find all the solutions of (3.1):

Solve[(1-u)^3/a^4+u^3/b^4= =1/(a^2-b^2)^2,u]

The three solutions for u are, $u = b^2/(b^2 - a^2)$, $u = b^2/(b^2 - a^2)$ and $u = b^2(2a^2 - b^2)/(a^4 - b^4)$. However, since $u = \sin^2\theta$ is nonnegative and $a > b$, the first two of these solutions (identical) are not possible. Therefore, the only *possible* solution is $u = b^2(2a^2 - b^2)/(a^4 - b^4)$. Clearly $u > 0$. However, since $u = \sin^2\theta$, u must be less than or equal to 1. One can easily show that $b^2(2a^2 - b^2)/(a^4 - b^4) \le 1$ if and only if $a^2 \ge 2b^2$. Therefore, the ellipse intersects its evolute if and only if $a^2 \ge 2b^2$. However, the eccentricity e of ellipse (1.1) is given by $e^2 = 1 - b^2/a^2$. Then it is easy to see that $a^2 \ge 2b^2$ if and only if $e \ge 1/\sqrt{2} \approx 0.7071$. This leads to the following proposition:

Proposition 3.1 Consider the ellipse (1.1) and its evolute (1.13).

(a) These curves intersect if and only if $e \ge 1/\sqrt{2}$.

(b) If $e = 1/\sqrt{2}$, then the two curves intersect at two distinct points $(0, \pm b)$. If $e > 1/\sqrt{2}$, then the two curves intersect at four distinct points P, Q, R and S, respectively in quadrants I-IV given by

$$\left(\pm a^2(a^2 - 2b^2)^{3/2}/\left((a^2 + b^2)\sqrt{a^4 - b^4}\right),\ \pm b^2(2a^2 - b^2)^{3/2}/\left((a^2 + b^2)\sqrt{a^4 - b^4}\right)\right).$$

(c) Under the case $e > 1/\sqrt{2}$, the points P, Q, R, S all lie on another ellipse if $e \ne e_0$, where $e_0 = \sqrt{-3 - \sqrt{13} + \sqrt{2}\sqrt{23 + 7\sqrt{13}}}/2 \approx 0.8968$. If $e > e_0$, the major axis of the new ellipse is parallel to that of the original ellipse, and has eccentricity $e_1 = \sqrt{(e^2 - 2)(e^8 + 3e^6 - 4e^4 + 2e^2 - 1)/(1 - 2e^2)^3}$. If $e < e_0$, the major axis of the new ellipse is perpendicular to that of the original ellipse, and has eccentricity $e_2 = \sqrt{(e^2 - 2)(e^8 + 3e^6 - 4e^4 + 2e^2 - 1)/(1 + e^2)^3/(1 - e^2)}$.

Proof. (a) follows directly from the paragraph just preceding Proposition 3.1.

(b) If $e = 1/\sqrt{2}$, then $a^2 = 2b^2$. However, since $u = \sin^2\theta = b^2(2a^2 - b^2)/(a^4 - b^4)$, one can easily show that $\cos^2\theta = a^2(a^2 - 2b^2)/(a^4 - b^4)$. Therefore, $\cos\theta = 0$, since $a^2 = 2b^2$. Then the equation (1.13) implies that $x = [(a^2 - b^2)\cos^3\theta]/a = 0$ and $y = [(a^2 - b^2)\sin^3\theta]/b = \pm b$. Therefore, if $e = 1/\sqrt{2}$, then the ellipse and its evolute intersect at two distinct points $(0, \pm b)$.

On the other hand, if $e > 1/\sqrt{2}$, then $a^2 > 2b^2$. The previous discussion implies that $\cos\theta = \pm a\sqrt{(a^2 - 2b^2)/(a^4 - b^4)}$, and $\sin\theta = \pm b\sqrt{(2a^2 - b^2)/(a^4 - b^4)}$.

Therefore, the equation (1.13) implies that the two curves intersect at four distinct points P, Q, R and S, whose x and y-coordinates are given by

$$x = \pm a^2(a^2 - 2b^2)^{3/2}/((a^2 + b^2)\sqrt{a^4 - b^4}) \qquad (3.2)$$
$$y = \pm b^2(2a^2 - b^2)^{3/2}/((a^2 + b^2)\sqrt{a^4 - b^4})$$

The following graph illustrates the relative positions of the ellipse and its evolute for $e > 1/\sqrt{2}$.

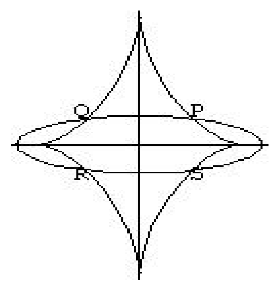

Fig. 3.1. Intersection of the Ellipse and its Evolute for $e > 1/\sqrt{2}$

(c) Suppose that $e > 1/\sqrt{2}$. Let us denote any of the intersections points P, Q, R or S by (u, v). The equations (3.2) imply

$$u^2/(a^4(a^2 - 2b^2)^3) + v^2/(b^4(2a^2 - b^2)^3) = 2/((a^2 + b^2)^2(a^4 - b^4)) \qquad (3.3)$$

The above equation (3.3) implies that all four points $\mathbf{p}, \mathbf{q}, \mathbf{r}$ and $\mathbf{s}$ lie on an ellipse or circle given by the following equation

$$\mathbf{x}^2/\mathbf{a}^2 + \mathbf{y}^2/\mathbf{b}^2 = 1 \tag{3.4}$$

where, the quantities $\mathbf{a}, \mathbf{b} > 0$ are given by

$$\begin{aligned} \mathbf{a}^2 &= 2a^4(a^2 - 2b^2)^3/((a^2+b^2)^2(a^4 - b^4)) \\ \mathbf{b}^2 &= 2b^4(2a^2 - b^2)^3/((a^2+b^2)^2(a^4 - b^4)) \end{aligned} \tag{3.5}$$

Depending on the eccentricity e of the original ellipse, $\mathbf{a}$ can be less than , greater than or equal to $\mathbf{b}$. Therefore, let us compute the ratio $\mathbf{b}/\mathbf{a}.$ The equations (3.5) imply that

$$\frac{\mathbf{b}}{\mathbf{a}} = (1 - e^2)\left(\frac{1 + e^2}{-1 + 2e^2}\right)^{3/2} \tag{3.6}$$

One can use *Mathematica* to show that $\mathbf{b}/\mathbf{a} = 1$ if and only if $e = e_0$ where

$$e_0 = \sqrt{-3 - \sqrt{13} + \sqrt{2}\sqrt{23 + 7\sqrt{13}}}\,/2 \approx 0.896 \tag{3.7}$$

Therefore, the equation (3.4) represents an ellipse if $e \neq e_0.$

One can further show that if $e > e_0,$ then $\mathbf{b} < \mathbf{a}.$ This implies that the major axis of the new ellipse (3.4) will be parallel to that of the original ellipse. The eccentricity $\mathbf{e}_1$ of the new ellipse is given by $\mathbf{e}_1^2 = 1 - \mathbf{b}^2/\mathbf{a}^2.$ Therefore, the equation (3.6) implies that

$$\mathbf{e}_1 = \sqrt{(e^2 - 2)(e^8 + 3e^6 - 4e^4 + 2e^2 - 1)/(1 - 2e^2)^3} \tag{3.8}$$

On the other hand, if $1/\sqrt{2} < e < e_0,$ then $\mathbf{b} > \mathbf{a}.$ This implies that the major axis of the new ellipse will be perpendicular to that of the original ellipse. The eccentricity $\mathbf{e}_2$ of the new ellipse is given by $\mathbf{e}_2^2 = 1 - \mathbf{a}^2/\mathbf{b}^2.$ Then the equation (3.6) implies that

$$\mathbf{e}_2 = \sqrt{(e^2 - 2)(e^8 + 3e^6 - 4e^4 + 2e^2 - 1)/(1 + e^2)^3/(1 - e^2)} \tag{3.9}$$

∎

We will use some of the information obtained in this section to answer a probability issue arising from normal lines to the ellipse.

4. Normal Lines, Probabilities and Elliptic Integrals

Consider the ellipse given by equation (1.1) with eccentricity $e > 1/\sqrt{2}.$ Suppose one chooses a point E on the ellipse at random. We are interested in calculating the probability that four normal lines can be drawn from E to the ellipse. We will first state and prove a lemma:

Lemma 4.1 Consider the ellipse given by equation (1.1) with eccentricity $e > 1/\sqrt{2}$. As done in the previous section, let A, B, C and D denote the distinct points of intersections of the ellipse and its evolute, respectively in the quadrants I-IV. Then the eccentric angle θ of the point A is given by $\tan\theta = \sqrt{(1-e^2)(1+e^2)^3/(2e^2-1)^3}$ where $0 < \theta < \pi/2$, and e denotes the eccentricity of the ellipse.

Proof. Since the point A lies on the ellipse, one can write it as $(a\cos\theta, b\sin\theta)$ where $0 < \theta < \pi/2$. By definition θ is called the eccentric angle of the point A (see [3]). However, the discussion prior to Proposition 3.1 implied that the point A can also be written as $((a^2-b^2)\cos^3\alpha/a, (a^2-b^2)\sin^3\alpha/b)$ where

$$\sin\alpha = b\sqrt{(2a^2-b^2)/(a^4-b^4)} \tag{4.1}$$

$$\cos\alpha = a\sqrt{(a^2-2b^2)/(a^4-b^4)} \tag{4.2}$$

Also refer to the proof of Proposition 3.1 (b). Then we have the following two equations:

$$a\cos\theta = (a^2-b^2)\cos^3\alpha/a \tag{4.3}$$

$$b\sin\theta = (a^2-b^2)\sin^3\alpha/b \tag{4.4}$$

One can divide equation (4.4) by equation (4.3) to obtain

$$\tan\theta = \frac{a^2}{b^2}\tan^3\varphi. \qquad (4.5)$$

However, the equations (4.1) and (4.2) imply that $\tan\varphi = (b/a)\sqrt{(2a^2 - b^2)/(a^2 - 2b^2)}$. Substituting this back in equation (4.5) and noting that $b^2/a^2 = 1 - e^2$, one can easily show that

$$\tan\theta = \sqrt{(1 - e^2)(1 + e^2)^3/(2e^2 - 1)^3} \qquad (4.6)$$

The above equation (4.6) defines the eccentric angle θ of the point Q, which is the point of intersection in the first quadrant, of the ellipse and its evolute. ∎

We are now in a position to answer the probability issue raised at the beginning of this section. Our calculations involve a certain type of elliptic integrals.

Theorem 4.1 Consider the ellipse given by equation (1.1) with eccentricity $e > 1/\sqrt{2}$. Suppose one chooses a point P on the ellipse at random. Then the probability p that one can draw four distinct normal lines from P to the ellipse is given by

$$p = \frac{\int_{\theta}^{\pi/2}\sqrt{1 - (e^2/(e^2 - 1))\sin^2 t}\,dt}{\int_{0}^{\pi/2}\sqrt{1 - (e^2/(e^2 - 1))\sin^2 t}\,dt} \qquad (4.7)$$

where θ is the eccentric angle of the point Q, as given in Lemma 4.1.

Proof. Since the eccentricity $e > 1/\sqrt{2}$, the relative positions of the ellipse and its evolute are given as in Fig. 3.1. In the introductory section we proved that the set of all the points on the plane, from where one can draw exactly four distinct normal lines to the ellipse is precisely the interior of the evolute of the ellipse (see also Fig. 1.3). However, the randomly chosen point X is *on* the ellipse. Therefore, there exists four distinct normals from X to the ellipse if and only if X lies on the portion of the ellipse strictly inside its evolute. Hence the required probability is given by the ratio L_1/L_2 where L_1 denotes the sum of the arc lengths of the ellipse between T and U, and between V and W, and L_2 denotes the total arc length of the ellipse. Therefore, using the arc length formula, one obtains (see [17])

$$P = \frac{\int_{\phi}^{\pi/2} \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\, d\theta}{\int_{0}^{\pi/2} \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\, d\theta} \qquad (4.8)$$

Using $\cos^2\theta = 1 - \sin^2\theta$, and $a^2/b^2 - 1 = e^2/(1 - e^2)$, one can further simplify equation (4.8) into equation (4.7). $\blacksquare$

The numerator and the denominator of the right-hand side of equation (4.7) are called elliptic integrals. They cannot be computed by hand even for a given specific value of e. However, one can conveniently use *Mathematica* to compute them for a given specific value of e. For example, the *Mathematica* command "**EllipticE[**ϕ, m**]**" computes the elliptic integral of the second kind $\int_0^{\phi} \sqrt{1 - m \sin^2\theta}\, d\theta$. The following example computes the probability given by equation (4.7) for $e = 0.85$ (Note: $0.85 > 1/\sqrt{2}$).

Example 4.1 Compute the probability given by equation (4.7) for $e = 0.85$.

Solution Here are the required *Mathematica* commands:

```
e=0.85;
phi=ArcTan[Sqrt[(1-e^2)(1+e^2)^3/(2e^2-1)^3]];
m = e^2/(e^2 - 1);
i1 = EllipticE[Pi/2, m];
i2 = EllipticE[phi, m];
(i1 - i2)/i1
```

The output is approximately !Þ"*(& . è

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