The Buffon needle problem revisited in a pedagogical perspective

Ivar Johannesen

Oslo University College
Faculty of Engineering
Cort Adelers gt.30 N–0254 Oslo, Norway
ivar.johannesen@iu.hio.no

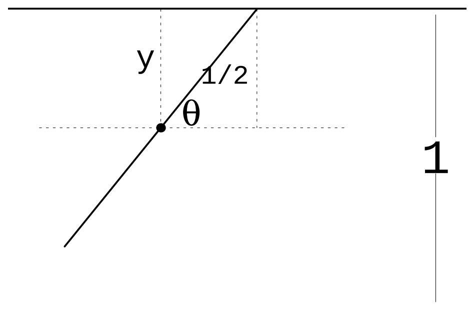
Imagine marking the floor with many equally spaced parallel lines and a thin stick whose length exactly equals the distance L = 1 between the lines. If the stick is thrown on the floor, the stick may or may not cross one of the lines. The probability for a hit will involve π. This is surprising since there are no circles involved, on the contrary all is typically linear. If we repeat the experiment many times, and keep track of the hits, we can get an estimate of the irrational number π.
We also consider sticks of length L > 1. This exercise can easily be done in a first year calculus course, where the students are challenged to consider concepts such as probability, definite integral, symmetry and inverse trigonometric function. The solution to this problem will therefore give many applications in a variety of fields in calculus.
We go on throwing regular polygons of different sizes, increasing the number of edges and at last reach the ultimate goal: throwing circular objects. This paper illustrates the process of throwing sticks, polygons and circles analytically and graphically, and carrry out calculations for different n − gons. The result always include the number π, except when the circle is introduced! We will also see the circle result as a limiting value when n increases to infinity.

■ Introduction

The problem of throwing sticks on a set of parallel equidistant lines was first raised by the French naturalist and mathematician Georges Louis Leclerc Comte de Buffon in 1733, and later solved in 1777 by Buffon himself. Despite the linearity of the situation, the result gives us a method to compute the irrational number π. For more than 250 years scientists have been intrigued by this puzzle, as can be seen by a quick search on the Internet. Many authors also extend the exercise to throwing regular polygons. In this paper I will consider regular polygons with n both even and odd. When the number of vertices is even, opposite vertices are situated on the diameter of the circumscribed circle. There are no diametrically opposed vertices in odd regular polygons, and therefore these n – gons offer more challenge for the students to handle. The length L of the stick is replaced by the diameter L of the circumscribed circle when regular polygons are considered.

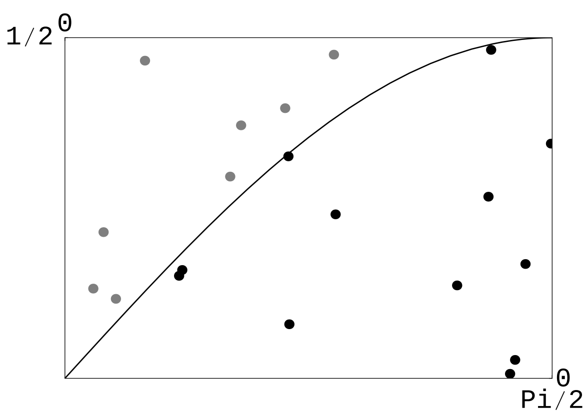
This paper illustrates the process of throwing sticks, polygons and circles analytically and graphically, and carrry out calculations for different n – gons. The mathematics necessary is elementary and suitable for students in a first calculus course. The students will solve the necessary integrals and calculate the probabilities by hand before invoking *Mathematica*.

The introductory part of the lab considers sticks of length L = 1, the same unit length as the distance between lines. The idea is described in [1], including relevant *Mathematica* code for illustrations. Each throw can be fully described by two parameters: the distance y from the center of the stick to nearest line, and the acute angle θ the stick makes with any parallell line.



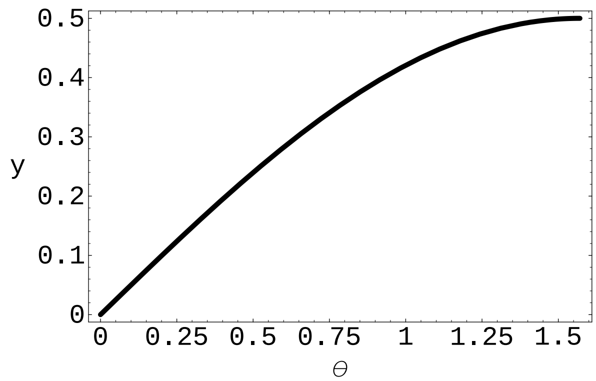
From the figure we see that the stick will hit the line if $y \leq \frac{1}{2}\sin\theta$. In the parameter space (θ, y) the graph of the function $y = \frac{1}{2}\sin\theta$ will be the border line between areas representing hits and misses. In the next figure the misses are drawn in gray and the hits in black. Due to symmetry we need only consider $0 \leq y \leq \frac{1}{2}$, $0 \leq \theta \leq \frac{\pi}{2}$.

From In[20]:=



Relating to the main topic in this paper, we regard the stick as a degenerate polygon with two vertices and reformulate our function expression according to this:

In[8]:=
```
Plot[1/2 Cos[π/2 - θ], {θ, 0, π/2},
  Frame → True, FrameLabel → {"θ", "y"},
  RotateLabel → False, PlotStyle → Thickness[0.01]];
```



The probability for hitting a line is the ratio of area under graph to area of parameter space.

In[60]:= $P[hit] = \dfrac{\int_0^{\frac{\pi}{2}} \frac{1}{2} \text{Cos}[\frac{\pi}{2} - \theta]\, d\theta}{\frac{\pi}{4}}$

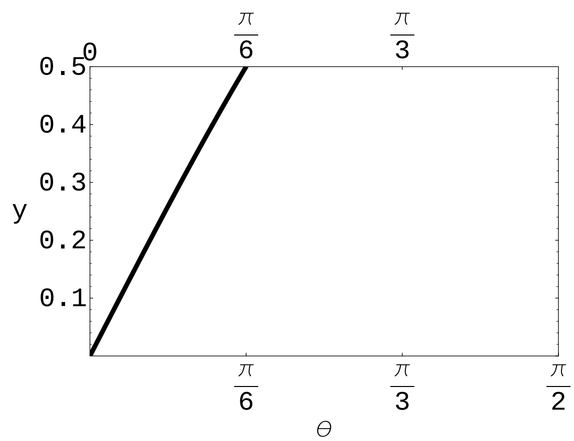
Out[60]= $\dfrac{2}{\pi}$

This result is interesting because it suggests a way to estimate the number π. Let a group of students draw parallel, equidistant lines on a large piece of paper and throw a substantial number of sticks on it, keeping record of the hits. If n needles hit a line out of t tries, then the students get an approximate number for Pi: $\pi \approx \frac{2t}{n}$

■ Long sticks

Let us look at sticks with arbitrary length L. When L≤ 1, the probability for hits is directly proportional to L. When L > 1, large values of θ will always give hits. The situation for L = 2 is illustrated below.

In[9]:=
```
Plot[Cos[π/2 - θ], {θ, 0, π/3}, PlotRange → {{0, π/2}, {0, 1/2}},
  Frame → True, FrameLabel → {"θ", "y"},
  RotateLabel → False, PlotStyle → Thickness[0.01],
  FrameTicks → {{0, π/6, π/3, π/2}, Automatic}];
```



For $\theta \geq \frac{\pi}{6}$ the stick of length 2 will always cross a line because it is too much inclined to avoid the line. On the other hand, arbitrary long sticks avoid hitting a line if the inclination is small enough. The probability for a stick of length L is given by the function probSticks:

In[60]:=
$$\text{probSticks}[L\_] = \frac{4}{\pi}\,\text{If}\Big[L < 1, \int_0^{\frac{\pi}{2}} \frac{L}{2}\,\text{Sin}[\theta]\,d\theta, \left(\frac{1}{2}\left(\frac{\pi}{2} - \text{ArcSin}\Big[\frac{1}{L}\Big]\right) + \int_0^{\text{ArcSin}[\frac{1}{L}]} \frac{L}{2}\,\text{Sin}[\theta]\,d\theta\right)\Big];$$

The expression is interesting for several reasons. First we have a "real" situation in which an inverse trigonometric function arises naturally. Second, the definite integral that makes up the last term is noteworthy in that finding an antiderivative is easy, while evaluating it at the integral's endpoints requires a little more work. The students are encouraged to simplify cos(arc sin $\frac{1}{L}$) and verify the simpler expression:

In[31]:=
$$\text{probSticks}[L\_] := \text{If}\Big[L \leq 1, \frac{2\,L}{\pi}, \frac{2}{\pi}\left(\left(L - \sqrt{L^2 - 1}\right) + \text{ArcCos}\Big[\frac{1}{L}\Big]\right)\Big]$$

We see that the probabilities always involve the factor $\frac{1}{\pi}$. For $L \leq 1$the graph is linear

In[32]:= `probSticks [1]`

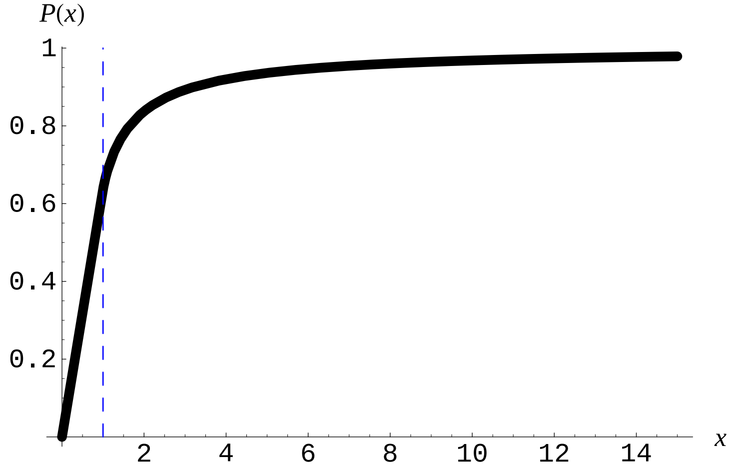
Out[32]= $\frac{2}{\pi}$

In[4]:= `N[%]`

Out[4]= 0.63662

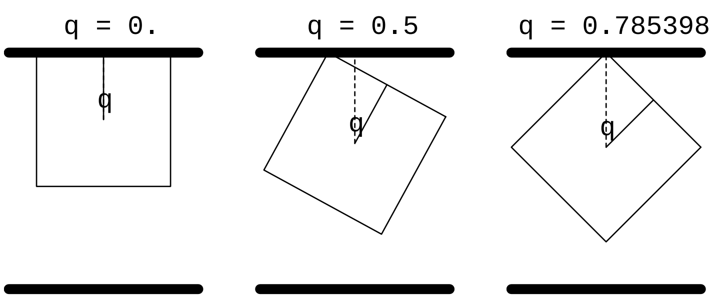
We summarize our results for sticks of any length by plotting the probability of hitting a line as a function of L.

In[40]:=
```
Plot[probSticks[L], {L, 0, 15}, PlotStyle → Thickness[0.015],
  Epilog → {Blue, Dashing[{.02}], Line[{{1, 0}, {1, 1}}]},
  AxesLabel -> TraditionalForm /@ {x, P[x]}];
```



■ Tossing squares

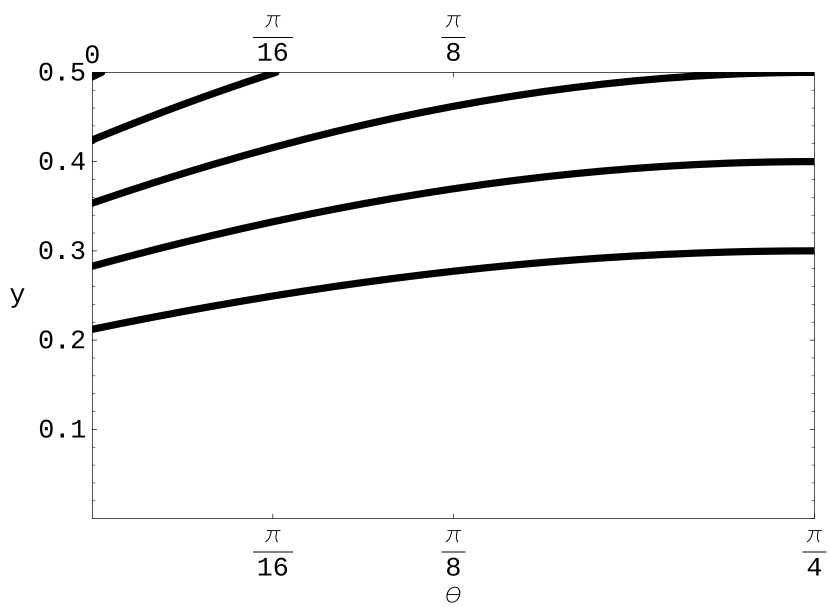
We start our investigation of regular polygons by tossing squares on the ruled floor. Let θ be the acute angle between the vertical and a line through square center and midpoint of an edge. Other choices for the angle is also possible. Figure 3 shows some constellations where the square just touches the line.



The graph in the parameter space dividing hits and misses is given by $y = \frac{L}{2}\cos(\frac{\pi}{4} - \theta)$, where L is the length of the square diagonal. This is the same as the diameter of the circumscribed circle. Due to symmetry it is enough to consider $0 \le \theta \le \frac{\pi}{4}$.

For $L \le 1$ we always have $y \le \frac{1}{2}$. What about squares whose diameter is greater than 1? Since the θ parameter is restricted to [0, Pi /4], we must consider the limit $L = \sqrt{2}$. If L increases beyond that value, there will always be hits with at least one side of the square. The curve dividing hits and misses, will exceed the value of 1/2 for all values of θ, and the plot in parameter space will be empty. When L > 1, the polygon will hit a line if $\frac{L}{2}\cos(\frac{\pi}{4} - \theta) < \frac{1}{2}$ and $\theta < \frac{\pi}{4}$. This means that $\theta > \frac{\pi}{4} - \arccos\left(\frac{1}{L}\right)$.

```
In[10]:= Show[Table[Plot[Evaluate[L/2 Cos[π/4 - θ]],
           {θ, 0, If[L ≤ 1, π/4, π/4 - ArcCos[1/L]]}, PlotRange →
           {{0, π/4}, {0, 1/2}}, Frame → True, FrameLabel → {"θ", "y"},
           RotateLabel → False, PlotStyle → Thickness[0.01],
           FrameTicks → {{0, π/16, π/8, π/4}, Automatic},
           DisplayFunction → Identity]], {L, 0.6, 1.4, 0.2}],
         DisplayFunction → $DisplayFunction];
```



The area in parameter space corresponding to hits will be $\int_0^{\frac{\pi}{4}} \frac{L}{2} \cos(\frac{\pi}{4} - \theta)\, d\theta = \int_0^{\frac{\pi}{4}} \frac{L}{2} \cos\theta\, d\theta$ for $L \le 1$, and $\frac{1}{2}\left(\frac{\pi}{4} - \left(\frac{\pi}{4} - \arccos\left(\frac{1}{L}\right)\right)\right) + \int_0^{\frac{\pi}{4} - \arccos\left(\frac{1}{L}\right)} \frac{L}{2} \cos(\frac{\pi}{4} - \theta)\, \theta =$

$$\frac{1}{2} \arccos\left(\frac{1}{L}\right) + \int_{\mathrm{arcCos}\left(\frac{1}{L}\right)}^{\frac{\pi}{4}} \frac{L}{2} \cos\, \theta\, d\theta$$

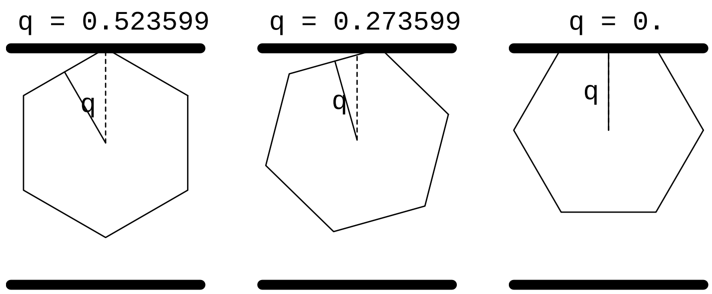
for $1 < L < \sqrt{2}$. This gives us the probability function

```
In[33]:= probSquare[L_] :=
           8/π Piecewise[{{L/2 ∫_0^(π/4) Cos[θ] dθ, 0 ≤ L ≤ 1}, {(1/2 ArcCos[1/L] +
             L/2 ∫_(ArcCos[1/L])^(π/4) Cos[θ] dθ), 1 < L ≤ √2}, {π/8, True}}]

In[34]:= {probSquare[1], probSquare[√2]}

Out[34]= {2 √2/π, 1}
```

In[35]:=
```
N[%]
```
Out[35]=
```
{0.900316, 1.}
```

■ Hexagons

Throwing hexagons follow the same outline as squares. Opposite vertices lie on the diameter of the circumscribed circle, and so we have symmetry about $\theta = \frac{\pi}{6}$.



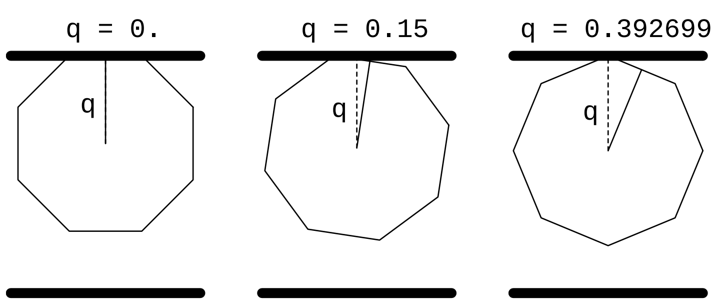
For $L > 1$ the hexagon will hit a line when $\theta \geq \frac{\pi}{6} - \arccos\left(\frac{1}{L}\right)$. For $L \geq \frac{2}{\sqrt{3}}$ at least one line will always be hit.

In[69]:=
```
probHexagon[L_] :=
  12/π Piecewise[{{L/2 ∫_0^(π/6) Cos[θ] dθ, 0 ≤ L ≤ 1}, {(1/2 ArcCos[1/L] +
      L/2 ∫_(ArcCos[1/L])^(π/6) Cos[θ] dθ), 1 < L ≤ 2/√3}, {π/12, True}}]
```

In[70]:=
```
{probHexagon[1], probHexagon[2/√3]}
```

Out[70]=
```
{3/π, 1}
```

■ Octagons



For $L > 1$ the octagon will hit a line when $\theta \geq \frac{\pi}{8} - \arccos\left(\frac{1}{L}\right)$. For $L \geq \frac{1}{\cos(\frac{\pi}{8})}$ at least one line will always be hit.

In[1]:= **probOctagon[L_] :=**

$$\frac{16}{\pi}\,\text{Piecewise}\Big[\Big\{\Big\{\frac{L}{2}\int_0^{\frac{\pi}{8}}\text{Cos}[\theta]\,d\theta,\ 0\le L\le 1\Big\},\ \Big\{\Big(\frac{1}{2}\,\text{ArcCos}\Big[\frac{1}{L}\Big]+\frac{L}{2}\int_{\text{ArcCos}[\frac{1}{L}]}^{\frac{\pi}{8}}\text{Cos}[\theta]\,d\theta\Big),\ 1<L\le\sqrt{4-2\sqrt{2}}\Big\},\ \Big\{\frac{\pi}{16},\ \text{True}\Big\}\Big\}\Big]$$

In[3]:= $\{\text{probOctagon}[1],\ \text{probOctagon}[\frac{1}{\text{Cos}[\frac{\pi}{8}]}]\}$

Out[3]= $\{\frac{8\,\text{Sin}[\frac{\pi}{8}]}{\pi},\ 1\}$

In[4]:= $\text{probOctagon}[1]\ /.\ \text{Sin}[x\_]\to\sqrt{\frac{1-\text{Cos}[2\,x]}{2}}\ //\ \text{Simplify}$

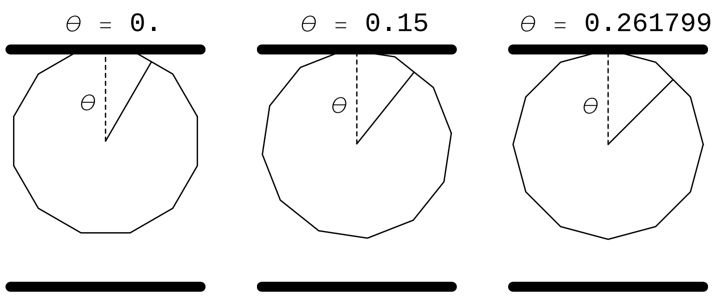
Out[4]= $\frac{4\sqrt{2-\sqrt{2}}}{\pi}$

In[5]:= **N[%]**

Out[5]= 0.974495

■ Dodecagons

From In[65]:=



In[6]:= **probDodecagon[L_] :=**

$$\frac{24}{\pi}\,\text{Piecewise}\Big[\Big\{\Big\{\frac{L}{2}\int_0^{\frac{\pi}{12}}\text{Cos}[\theta]\,d\theta,\ 0\le L\le 1\Big\},\ \Big\{\Big(\frac{1}{2}\,\text{ArcCos}\Big[\frac{1}{L}\Big]+\frac{L}{2}\int_{\text{ArcCos}[\frac{1}{L}]}^{\frac{\pi}{12}}\text{Cos}[\theta]\,d\theta\Big),\ 1<L\le 2\sqrt{2-\sqrt{3}}\Big\},\ \Big\{\frac{\pi}{24},\ \text{True}\Big\}\Big\}\Big]$$

In[7]:= $\{\text{probDodecagon}[1],\ \text{probDodecagon}[\frac{1}{\text{Cos}[\frac{\pi}{12}]}]\}$

Out[7]= $\{\frac{3\sqrt{2}\,(-1+\sqrt{3})}{\pi},\ 1\}$

In[59]:= **N[%]**

Out[59]= {0.988616, 1.}

■ 2n − gons

For higher order n − gons where n is even we encounter the same sort of symmetry about $\theta = \frac{\pi}{n}$ and we always get a hit when $L \geq \frac{1}{\cos(\frac{\pi}{n})}$.

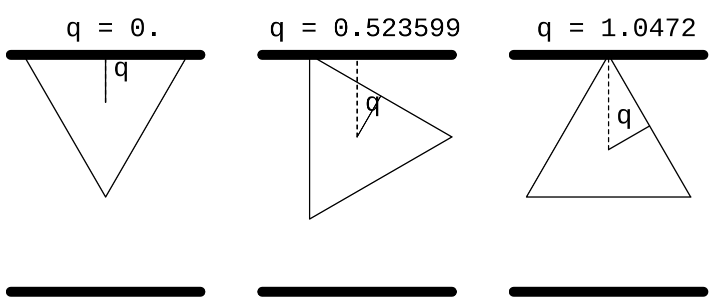
In[8]:=
```
probNgon[n_?EvenQ, L_] :=
 (2 n/π) Piecewise[{{(L/2) ∫_0^(π/n) Cos[θ] ⅆθ, 0 ≤ L ≤ 1},
   {((1/2) ArcCos[1/L] + (L/2) ∫_(ArcCos[1/L])^(π/n) Cos[θ] ⅆθ), 1 < L ≤ 1/Cos[π/n]},
   {π/(2 n), True}}]
```

In[11]:=
```
{probNgon[20, 1], probNgon[20, 1/Cos[π/20]]} (* icosagon *)
```

Out[11]=
$$\left\{\frac{20\,\mathrm{Sin}\left[\frac{\pi}{20}\right]}{\pi},\ 1\right\}$$

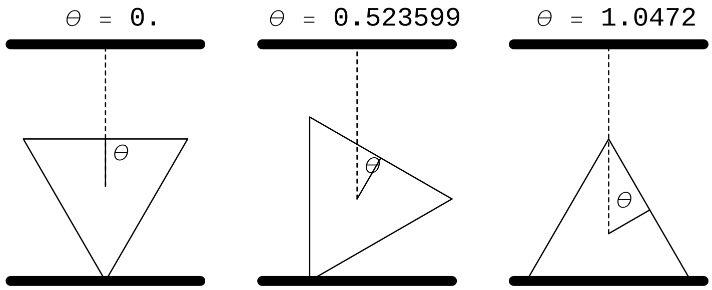
■ Tossing triangles

Next we look at equilateral triangles. Considering regular odd polygons, the adjacent vertices do not lie on the diameter of the circumscribed circle. This means that we must take the full distance between lines into consideration.



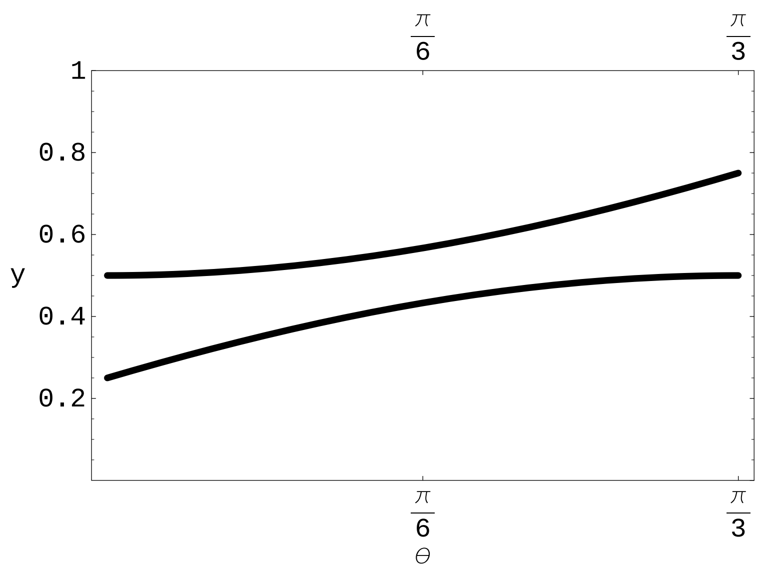
Let y be the vertical distance from the top line to the center of the triangles. This is where the medians cross each other, and also the heights since the triangle is regular. From figure 7 we see that the border line between hit area and miss area is $y = \frac{L}{2}\cos(\frac{\pi}{3} - \theta)$, where L is the diameter of the circumscribed circle. This means that the heights (medians) has length $\frac{3L}{4}$. For $L > \frac{4}{3}$the triangle therefore has to cut one or more lines. But there is another border line, as figur 8 shows:

From In[6]:=



For $y > 1 - \frac{L}{2}\cos(\theta)$ the triangles will cut the lower line. So the hit area consists of two distinct parts in parameter space:

In[11]:=
```
Plot[{1/2 Cos[π/3 - θ], 1 - 1/2 Cos[θ]},
  {θ, 0, π/3}, Frame → True, FrameLabel → {"θ", "y"},
  RotateLabel → False, PlotStyle → Thickness[0.01],
  FrameTicks → {{π/6, π/3}, Automatic}, PlotRange → {0, 1}] ;
```



The total hit area in parameter space is therefore:

$$\int_0^{\frac{\pi}{3}} \frac{L}{2}\cos\left(\frac{\pi}{3}-\theta\right) d\theta + \int_0^{\frac{\pi}{3}} \left(1 - \left(1 - \frac{L}{2}\cos\theta\right)\right) d\theta =$$

$$\int_0^{\frac{\pi}{3}} \frac{L}{2}\cos\left(\frac{\pi}{3}-\theta\right) d\theta + \int_0^{\frac{\pi}{3}} \frac{L}{2}\cos\theta\, d\theta = \int_0^{\frac{\pi}{3}} L\cos\theta\, d\theta$$

We see that the two separate areas are equal, this simplifies the probability calculation .

In[22]:=
```
probTriangle[1] = 3/π ∫_0^(π/3) Cos[θ] dθ
```

Out[22]= $\frac{3\sqrt{3}}{2\pi}$

```
In[23]:= N[%]
Out[23]= 0.826993
```

In[4]:= $\texttt{y[x\_, L\_] :=}\ \frac{L}{2}\left(\text{Cos}\left[\frac{\pi}{3}-x\right]+\text{Cos}[x]\right)$

In[5]:= $\left\{\text{Solve}\left[y\left[\frac{\pi}{6}, L\right]==1, L\right], \text{Solve}\left[y\left[\frac{\pi}{3}, L\right]==1, L\right]\right\}$ // Flatten

Out[5]= $\left\{L\to\frac{2}{\sqrt{3}}, L\to\frac{4}{3}\right\}$

When $\frac{2}{\sqrt{3}} < L < \frac{4}{3}$, the triangle will cut the line when $\theta > \theta_1$, where θ_1 is the solution of the equation $\frac{L}{2}(\cos(\frac{\pi}{3}-\theta)+\cos\theta) = 1$, given $0 < \theta < \frac{\pi}{6}$. For $L \geq \frac{4}{3}$the triangle has to cross at least one line since then $\theta_1 \leq 0$. Notice that $\frac{4}{3} = \left(\frac{2}{\sqrt{3}}\right)^2$.

In[28]:= $\text{probTriangle[L\_] := If}\left[L\le\frac{2}{\sqrt{3}}, \frac{3}{\pi}\int_0^{\frac{\pi}{3}} L\,\text{Cos}[x]\,dx\right]$

In[30]:= $\left\{\text{probTriangle[1]}, \text{probTriangle}\left[\frac{2}{\sqrt{3}}\right]\right\}$

Out[30]= $\left\{\frac{3\sqrt{3}}{2\pi}, \frac{3}{\pi}\right\}$

■ Pentagons

The calculations follow the same outline as for triangles:

In[1]:= $\texttt{y[x\_, L\_] :=}\ \frac{L}{2}\left(\text{Cos}\left[\frac{\pi}{5}-x\right]+\text{Cos}[x]\right)$

In[3]:= $\left\{\text{Solve}\left[y\left[\frac{\pi}{10}, L\right]==1, L\right], \text{Solve}[y[0, L]==1, L]\right\}$ // Flatten

Out[3]= $\left\{L\to 2\sqrt{\frac{2}{5+\sqrt{5}}}, L\to\frac{8}{5+\sqrt{5}}\right\}$

If $L \geq \frac{8}{5+\sqrt{5}}$, the pentagon has to cut one line. Again we see that this limit is the square of the lower limit, as was the case with $n = 3$. For $\sqrt{\frac{8}{5+\sqrt{5}}} < L < \frac{8}{5+\sqrt{5}}$., there will always be hits if $\theta > \theta_1$, where θ_1 is the solution to the equation $\frac{L}{2}(\cos(\frac{\pi}{5}-\theta)+\cos\theta) = 1$, given $0 < \theta < \frac{\pi}{10}$.

In[121]:= $\text{probPentagon[L\_] := If}\left[L\le\sqrt{\frac{8}{5+\sqrt{5}}}, \frac{5}{\pi}\int_0^{\frac{\pi}{5}} L\,\text{Cos}[x]\,dx\right]$

In[132]:= $\left\{\text{ToRadicals[probPentagon[1]]}, \text{probPentagon}\left[\sqrt{\frac{8}{5+\sqrt{5}}}\right]\right\}$

Out[132]= $\left\{\frac{5\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}}{2\pi}, \frac{5\left(-1+\sqrt{5}\right)}{2\pi}\right\}$

■ 2n+1 – gons

In[39]:=
```
y[x_, L_] := L/2 (Cos[π/n - x] + Cos[x])
```

Value of L where all polygons in the most symmetric position will hit a line:

In[134]:=
```
Solve[y[π/(2 n), L] == 1, L]
```

Out[134]= $\{\{L \to \mathrm{Sec}[\frac{\pi}{2n}]\}\}$

Value of L where every polygon hits a line, independent of rotation:

In[135]:=
```
Solve[y[0, L] == 1, L]
```

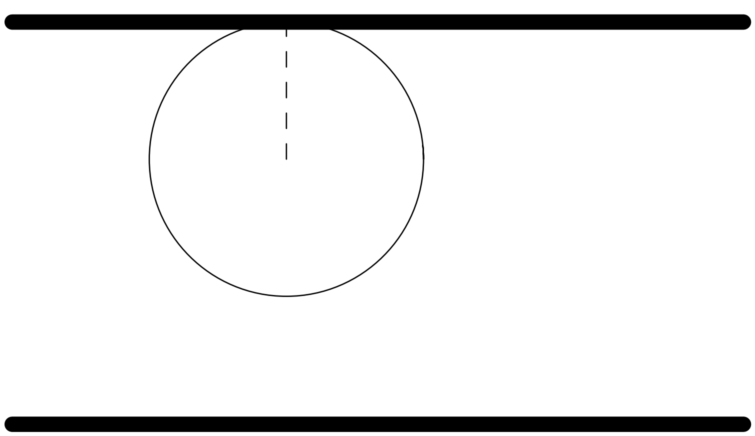
Out[135]= $\{\{L \to \frac{2}{1 + \mathrm{Cos}[\frac{\pi}{n}]}\}\}$

For higher order n – gons where n is odd we always hit a line when $L \geq \frac{2}{1+\mathrm{Cos}(\frac{\pi}{n})} = \left(\frac{1}{\cos(\frac{\pi}{2n})}\right)^2$. For $L \leq \frac{1}{\cos(\frac{\pi}{2n})}$ the probability is proportional to L, and for $\frac{1}{\cos(\frac{\pi}{2n})} \leq L \leq \left(\frac{1}{\cos(\frac{\pi}{2n})}\right)^2$ there will always be a hit if $\theta \geq \theta_1$, where θ_1 is a solution of the equation $\frac{L}{2}(\cos(\frac{\pi}{n} - \theta) + \cos\theta) = 1$, given $0 < \theta < \frac{\pi}{2n}$

■ Tossing coins

Suppose a penny with diameter L is thrown on a ruled surface. The position of the coin is independent of the angle θ, and the circle will hit the line whenever $y \leq \frac{L}{2}$. The border line dividing hits from misses is the straight line $y = \frac{L}{2}$.

From In[144]:=



In this case the parameter space is 1– dimensional, but we can define a random value θ_{max} and create a rectangle $[0,\theta_{max}]$ x $[0,\frac{1}{2}]$. The probability for hitting a line is therefore $\frac{L/2}{1/2} = L$, that is directly proportional to L when $L \leq 1$. The next plot illustrate the situation when $L = \frac{3}{4}$.

In[12]:=
```
Plot[3/8, {θ, 0, π/2}, Frame → True, FrameLabel → {"θ", "y"},
  RotateLabel → False, PlotStyle → Thickness[0.01],
  PlotRange → {0, 0.5}, FrameTicks → {{0, π/2}, {0, 0.375, 0.5}}];
```

In[148]:=
```
probCircle[L_] := If[L < 1, L, 1]
```

In[29]:=
```
{probCircle[3/4], probCircle[1]}
```

Out[29]= $\left\{\frac{3}{4}, 1\right\}$

For the first time, when the circle actually appears on the scene, the result does not involve π!

■ Summary for the case L = 1

In this paper we have extended the Buffon needle problem to include polygons thrown on a ruled floor, and calculated the probabilities for hits for various values of the diameter L of the circumscribed circle. Each time the answer involved the irrational nuimber π, and therefore indicated a simulation to estimate the value of this famous number. For $L = 1$ we summarize the results (the stick counts as 2− gon):

n= 2: $p = \frac{2}{\pi}$ N[p] = 0.63662

n= 3: $p = \frac{3\sqrt{3}}{\pi}$ N[p] = 0.82699

n = 4: $p = \frac{2\sqrt{2}}{\pi}$ N[p] = 0.90032

n = 5: $p = \frac{5\sqrt{\frac{1}{2}(5-\sqrt{5})}}{\pi}$ N[p] = 0.93549

n= 6: $p = \frac{3}{\pi}$ N[p] = 0.95493

n= 8: $p = \frac{4\sqrt{2-\sqrt{2}}}{\pi}$ N[p]= 0.97450

n = 12: $p = \frac{3\sqrt{2}\,(\sqrt{3}-1)}{\pi}$ N[p] = 0.98862

n = ∞: p = 1 N[p]= 1.0000

For each value of n we find $p(n) = \frac{n\sin(\frac{\pi}{n})}{\pi} = \frac{\sin(\frac{\pi}{n})}{\frac{\pi}{n}}$, and therefore $\lim_{n\to\infty} p(n) = 1$.
Increasing the number of vertices in the regular polygon to infinity, we therefore reach the result for tossing circles on the ruled floor.

For even n– gons we found the border line to be $y = \frac{L}{2}\cos(\frac{\pi}{n} - \theta)$ for $0 \le \theta \le \frac{2\pi}{n}$. All n–gons would cut a line if $L \ge \frac{1}{\cos(\frac{\pi}{n})}$.When n→∞ this gives $y = \frac{L}{2}$ independent of θ and always hits when $L \ge 1$. This is in agreement with the circular case.
For odd n – gons we found $y = \frac{L}{2}(\cos(\frac{\pi}{n} - \theta) + \cos\theta)$, $0 < \theta < \frac{\pi}{n}$. All n– gons would cut a line when $L \ge \frac{2}{\cos(\frac{\pi}{n}) + 1}$. As n→∞, this again is in accordance with the circles.
So the result of throwing pennies is fully compatible with the limiting results obtained by studying n – gons for large n.

In the Mathematica code we had to take into consideration the parity of n when calculating the border line function, but the probability function for $L = 1$ could be simplified to one simple formula for all n..

```
In[39]:= y[n_?EvenQ][x_, L_] := L/2 (Cos[π/n - x]) /; 0 ≤ x ≤ π/n

In[39]:= y[n_?OddQ][x_, L_] := L/2 (Cos[π/n - x] + Cos[x]) /; 0 ≤ x ≤ π/n

In[16]:= probNgon[n_] := n/π ∫_0^(π/n) Cos[θ] dθ
```

■ References

[1] Stan Wagon , Ed Packel: Animating Calculus, TELOS/ Springer Verlag, Santa Clara, 1996