Convergence improvement of infinite series by linear fractions

Shigeki Matsumoto

Konan University
Department of Information Science and Systems Engineering, Faculty of Science and Engineering, Konan University
8-9-1 Okamoto, Higashinada-ku, Kobe 658–8501, Japan

shigeki@konan-u.ac.jp

Abstract

Though the Gregory-Leibniz series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \) converges slowly to \( \pi/4 \), a linear (red) fraction of \( n \) accelerates the speed of convergence as follows;

\[ \text{In}[3]= \text{Table}[N[\{4 \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1}, \frac{(-1)^{n}}{n} + 4 \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1}\}], 20], \{n, 1000, 1005\}] // \text{TableForm} \]

Joseph Roy North observed an analogous phenomenon with respect to the truncated value of the series in 1988. In the present article we explain the essence of the phenomenon in an elementary and general method, and generate linear fraction terms in order to accelerate the convergence speed for various infinite series by virtue of Mathematica commands. For example the following acceleration by linear fractions for

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Joseph Roy North observed an analogous phenomenon with respect to the truncated value of the series in 1988. In the present article we explain the essence of the phenomenon in an elementary and general method, and generate linear fraction terms in order to accelerate the convergence speed for various infinite series by virtue of Mathematica commands. For example the following acceleration by linear fractions for \( \zeta(2) \) almost equals L.Euler’s ingenious acceleration

\[
H_2 \approx \log^2 2 + \sum_{n=1}^{\infty} \frac{1}{2n-1} \text{n}^2 = 3 \frac{1}{60} \left( \frac{1}{n-2} + \frac{1}{n+3} \right) - \frac{2}{15} \left( \frac{1}{n-1} + \frac{1}{n+2} \right) + \frac{37}{60} \left( \frac{1}{n+1} - \frac{1}{n} \right) + \sum_{k=1}^{n} \frac{1}{k^2};
\]

\[
N[\{f[20], \text{Zeta}[2], \text{Log}[2]^2 + \sum_{n=1}^{20} \frac{1}{2n-1} \text{n}^2 \}, 12] // TableForm
\]

\[
\begin{array}{l}
1.6493407030 \\
1.6493406685 \\
1.6493406287
\end{array}
\]

The following is another example of acceleration by linear fraction terms. A few fraction terms make the approximation accurate.

\[
g[n_] := \frac{1}{180} \left( \frac{1}{n-1} - \frac{1}{n+2} \right) + \frac{11}{60} \left( \frac{1}{n+1} - \frac{1}{n} \right) + \sum_{k=1}^{n} \frac{1}{k} - \frac{\text{Log}[n] + \text{Log}[n+1]}{2};
\]

\[
N[\{g[20], \text{EulerGamma}\}, 12] // TableForm
\]

\[
\begin{array}{l}
0.577215665977 \\
0.577215664902
\end{array}
\]

**Goldbach’s method**

In 1729 Goldbach had an inequality estimation for \( \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} \),

\[
\sum_{k=1}^{\infty} \frac{1}{(k - \frac{7}{16})(k + \frac{9}{16})} + \sum_{k=1}^{6} \frac{1}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2};
\]

that is,

\[
\frac{41423}{25200} < \sum_{k=1}^{\infty} \frac{1}{k^2} < \frac{76997}{46800}.
\]
Applying his method to more general situation, we can improve the convergence of infinite series. That is, if the n-th partial sum of $\sum_{k=1}^{\infty} a_k$ is revised by the term $\sum_{k=n+1}^{\infty} b_k$, the convergence speed of a sequence \[ \{ \sum_{k=n+1}^{\infty} b_k + \sum_{k=1}^{n} a_k \} \] is equal to that of \[ \{ \sum_{k=1}^{\infty} (a_k - b_k) \} \].

\[
\sum_{k=n+1}^{\infty} b_k + \sum_{k=1}^{n} a_k = \\
\sum_{k=1}^{\infty} b_k + \sum_{k=1}^{n} (a_k - b_k)
\]

(3)

In the case of $a_k = \frac{1}{k^2}$, $b_k = \frac{1}{k^2 - \frac{1}{4}}$ for the above, we observe that the linear fraction of n,

\[
\sum_{k=n+1}^{\infty} \frac{1}{k^2 - \frac{1}{4}}
\]

improves the convergence speed of $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k^2}$.

\[
\text{Table} \left[ \left\{ \frac{2}{1 + 2n}, \zeta[2], 15 \right\}, \{n, 30, 40\} \right]
\]

\[
\text{TableForm}
\]

\[
\begin{array}{cc}
1.64493700284750 & 1.64493406684823 \\
1.64493673207396 & 1.64493406684823 \\
1.64493649359716 & 1.64493406684823 \\
1.6449362724213 & 1.64493406684823 \\
1.6449360562296 & 1.64493406684823 \\
1.6449359289170 & 1.64493406684823 \\
1.64493578011944 & 1.64493406684823 \\
1.6449356470206 & 1.64493406684823 \\
1.64493552678504 & 1.64493406684823 \\
1.64493541870314 & 1.64493406684823 \\
1.64493532103163 & 1.64493406684823 \\
\end{array}
\]

We give another example of acceleration of the convergence speed of $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k^2}$.
\[
\text{Series}\left[\frac{1}{n^2} - \left(\frac{p}{(n-1)(n+1)} + \frac{q}{(n-2)(n+2)} + \frac{r}{(n-3)(n+3)}\right), \{n, \infty, 8\}\right]
\]

\[\text{Out[1]} = H_1 - p - q - r L J 1 \text{E}€€€€€ n 2 + H - p - q - r L J 1 \text{E}€€€€€ n 4 + H - p - q - r L J 1 \text{E}€€€€€ n 6\]

\[\text{Out[2]} = \left\{\left\{p \to \frac{3}{2}, q \to \frac{3}{5}, r \to \frac{1}{10}\right\}\right\}\]

The finite sum of linear fractions,

\[\sum_{k=n+1}^{\infty} \left(\frac{p}{(k-1)(k+1)} + \frac{q}{(k-2)(k+2)} + \frac{r}{(k-3)(k+3)}\right) / . \text{values[[1]] // Apart}\]

\[\text{Out[3]} = \frac{1}{60 \left(-2 + n\right)} - \frac{2}{15 \left(-1 + n\right)} + \frac{37}{60 \ n} + \frac{37}{60 \ (1 + n)} - \frac{2}{15 \ (2 + n)} + \frac{1}{60 \ (3 + n)}\]

improves the convergence speed of \(\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k^2}\) much more.

\[\text{Out[3]/TableForm} = \{1.64493406705963, 1.64493406684823, 1.64493406684823, 1.64493406684823, 1.64493406684823, 1.64493406684823, 1.64493406684823, 1.64493406684823, 1.64493406684823, 1.64493406684823\}\]

\[\text{EulerGamma}\]

For Euler’s constant \(\lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right)\) we utilize the following equality.
\textbf{Convergence improvement of infinite series by linear fractions}

\textbf{In[57]} \quad \textbf{FullSimplify} \left[ \sum_{k=1}^{n} \frac{1}{k} - \frac{\text{\texttt{Log}}[n] + \text{\texttt{Log}}[n + 1]}{2} \right] =

= 1 - \frac{1}{2} \text{\texttt{Log}}[2] + \sum_{k=2}^{n} \left( \frac{1}{k} + \frac{1}{2} \right) \left( \frac{1 - \frac{1}{k}}{1 + \frac{1}{k}} \right), \quad \text{Assumptions} \to \{ n > 0, n \in \text{\texttt{Integers}} \}\]

\textbf{Out[57]} \quad \text{True}

We can find linear fraction terms for the acceleration of convergence by the same method.

\textbf{In[58]} \quad \textbf{Series} \left[ \frac{1}{k} + \frac{1}{2} \frac{\text{\texttt{Log}}}{1 + \frac{1}{k}} \right] -

= \left( \frac{s}{(k-1) k (k+1)} + \frac{t}{(k-2) k (k+2)} \right), \quad \{ k, \infty, 8 \} \]

\textbf{Out[58]} \quad \left( -\frac{1}{3} - s - t \right) \left( \frac{1}{k} \right)^{3} + \left( -\frac{1}{5} - s - 4 t \right) \left( \frac{1}{k} \right)^{5} + \left( -\frac{1}{7} - s - 16 t \right) \left( \frac{1}{k} \right)^{7} + O \left( \frac{1}{k} \right)^{9}

\textbf{In[59]} \quad \textbf{val} = \textbf{Solve} \left[ \left\{ -\frac{1}{3} - s - t = 0, -\frac{1}{5} - s - 4 t = 0 \right\}, \{ s, t \} \right]

\textbf{Out[59]} \quad \left\{ \left\{ s \to -\frac{17}{45}, t \to \frac{2}{45} \right\} \right\}

\textbf{In[60]} \quad \textbf{\texttt{\textbackslash .val[[1] /\ Apart}} \left[ \sum_{k=n+1}^{n} \left( \frac{s}{(k-1) k (k+1)} + \frac{t}{(k-2) k (k+2)} \right) \right]

\textbf{Out[60]} \quad \frac{1}{180} (-1 + n) - \frac{11}{60} n + \frac{11}{60} (1 + n) - \frac{1}{180} (2 + n)

\textbf{Reference}